SF2972 GAME THEORY
Lecture 2: Nash equilibrium

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26 January 2015

1 Normal form games

- The normal form of finite two-player games is usually represented as a bimatrix, with each entry being the associated payoff pair

- For games with more than two players and/or large (even infinite) strategy sets, we need a more general representation!

Definition 1.1 *A* **normal-form game** *is a triplet* $G = \langle I, S, u \rangle$ *where*

(a) I *is the set of* **players**

(b) $S = \times_{i \in I} S_i$ *is the set of* **strategy profiles**, $s = (s_i)_{i \in I} \in S$, *and* S_i *is the* **strategy set** *of player* i

(c) $u : S \to \mathbb{R}^{|I|}$ *is the* **combined payoff function**, *where* $u_i(s) \in \mathbb{R}$ *the* **payoff** *to player* i *when strategy profile* s *is played*

- Such a game is called *finite* if S is finite. (Here $|I|$ denotes the cardinality of the set I, the number of players, which may be finite or infinite.)

- From an abstract mathematical viewpoint, a normal-form game is any *function*

$$u : \times_{i\in I} S_i \to \mathbb{R}^{|I|}$$

- We do not here need to distinguish between "mixed" and "pure" strategies: let S be the relevant set of strategy profiles, pure or mixed or of whatever kind

- Notation: Given any strategy profile $(s_i)_{i \in I} \in S$ and any strategy $s_i' \in S_i$: write $\left(s_i', s_{-i}\right) \in S$ for the strategy profile in which player i uses strategy s_i' while all others $(j \neq i)$ play according to s

Definition 1.2 *A strategy profile $s^* \in S$ is a* **Nash equilibrium** *(***NE***) of $G = \langle I, S, u \rangle$ if*

$$u_i\left(s^*\right) \geq u_i\left(s_i, s_{-i}^*\right) \quad \forall i \in I, s_i \in S_i$$

- Notation: For any set X and function $f : X \to \mathbb{R}$ write

$$\arg\max_{x \in X} f(x) = \{x^* \in X : f(x^*) \geq f(x) \ \ \forall x \in X\}$$

The (potentially empty) set of *maximands* of f in X

- Suppose $I = \{1, 2, ..., n\}$. Then s^* is a Nash equilibrium iff ("iff" = "if and only if")

$$\begin{cases} s_1^* \in \arg\max_{s_1 \in S_1} u_1\left(s_1, s_2^*, s_3^*, ..., s_n^*\right) \\ s_2^* \in \arg\max_{s_2 \in S_2} u_2\left(s_1^*, s_2, s_3^*, ..., s_n^*\right) \\ ... \\ s_n^* \in \arg\max_{s_n \in S_n} u_n\left(s_1^*, s_2^*, ..., s_{n-1}^*, s_n\right) \end{cases}$$

- In other words, each player's strategy is then a *best reply* to the others strategies:

$$s_i^* \in \beta_i(s^*) := \text{arg} \max_{s_i \in S_i} u_i\left(s_i, s_{-i}^*\right) \quad \forall i \in I$$

- Here β_i is player i's *best-reply correspondence*, which to any given strategy profile $s \in S$ assigns i's (potentially empty) set of best replies

- Does every game have a Nash equilibrium? Can a game have more than one equilibrium?

- Challenge: To specify a list of (general and transparent) conditions that together guarantee the existence of at least one NE!

2 Existence of Nash equilibrium

- The following result generalizes Nash's (1950) existence result:

Theorem 2.1 (Glicksberg) *Let* $G = \langle I, S, u \rangle$ *be a game in which* $I = \{1, 2, ..., n\}$ *for some positive integer* n*,* $u : S \to \mathbb{R}^n$ *is continuous, and each strategy set* S_i *is a non-empty, compact and convex set in some Euclidean space. If, moreover, each payoff function* u_i *is quasi-concave in* $s_i \in S_i$ *(for any given* $s_{-i} \in S_{-i}$*) then there exists a Nash equilibrium.*

[The requirement that the strategy sets live in Euclidean spaces can be relaxed to the much weaker hypothesis that they live in locally convex Hausdorff spaces, see e.g. Aliprantis and Border, *Infinite-Dimensional Analysis*, 3rd et. (2006)]

3 Background mathematics

Notation: $\mathbb{N}$ the positive integers, $\mathbb{R}$ the reals, $\mathbb{R}_+$ the non-negative reals

- **Set properties** you are supposed to know**:** *empty*, *finite, open*, *closed*, *bounded*, *compact*, *convex*

- The *upper-contour sets* of any function $f : X \to \mathbb{R}$ is the collection of subsets

$$X(a) = \{x \in X : f(x) \geq a\}$$

for all $a \in \mathbb{R}$. The "contour map" of the function (cf. topographical maps in geography)

Definition 3.1 *A function $f : X \to \mathbb{R}$ with X convex is* **quasi-concave** *if all its upper contour-sets are convex.*

Proposition 3.1 *If $X \subseteq \mathbb{R}^n$ is convex and $f : X \rightarrow \mathbb{R}$ quasi-concave, then $X^* = \arg\max_{x \in X} f(x)$ is convex.*

Proof: If X^* is empty or contains only one point, then it is convex. If it contains more than one point, say x^* and x^{**}, then it must also contain all points $x = \lambda x^{**} + (1 - \lambda) x^*$ $(0 \leq \lambda \leq 1)$ between these. **Q.E.D.**

Theorem 3.2 (Weierstrass' Maximum Theorem) *If $X \subset \mathbb{R}^n$ is non-empty and compact, and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous, then $X^* = \arg\max_{x \in X} f(x)$ is non-empty and compact.*

Definition 3.2 *For any set* X *and function* $f : X \to X$ *from* X *to itself, a* **fixed point** *under* f *is any* $x^* \in X$ *such that* $x^* = f(x^*)$.

Theorem 3.3 (Brouwer's Fixed Point Theorem) *If* $X \subset \mathbb{R}^n$ *is non-empty, compact and convex, and* $f : X \to X$ *is continuous, then* f *has at least one fixed point.*

Given any sets X and Y, a correspondence from X to Y is any "rule" that to each $x \in X$ assigns a *non-empty subset* of Y. Formally:

Definition 3.3 *Given any sets* X *and* Y *a* **correspondence** $\varphi : X \rightrightarrows Y$ *is a function from* X *to the power set* 2^Y *of* Y *(the set of all subsets of* Y*) such that* $\varphi(x) \neq \varnothing \ \forall x \in X$.

Definition 3.4 *Given any set* X *and any correspondence* $\varphi : X \rightrightarrows X$ *from* X *to itself, a* **fixed point** *under* φ *is any* $x^* \in X$ *such that* $x^* \in \varphi(x^*)$.

Theorem 3.4 (Kakutani's Fixed-Point Theorem) *If $X \subset \mathbb{R}^n$ is non-empty, compact and convex, and $\varphi : X \rightrightarrows X$ is convex-valued, compact-valued and upper hemi-continuous, then φ has at least one fixed point.*

Definition 3.5 *A correspondence $\varphi : X \rightrightarrows Y$ is* **upper hemi-continuous (u.h.c.) at** *$x \in X$ if there for every open set B such that $\varphi(x) \subset B$ exists an open set A containing x such that $\varphi(x') \subset B\ \forall x' \in A \cap X$. A correspondence is* **upper hemi-continuous** *if it is is u.h.c. at all points in its domain X.*

3.1 Proof of Theorem 2.1

We are now in a position to prove Theorem 2.1!

1. For each player i, let $\beta_i : S \rightrightarrows S_i$ be the player's best-reply correspondence

2. By Weierstrass' maximum theorem: each $\beta_i(s)$ is non-empty and compact

3. By quasi-concavity and the convexity of S_i: each $\beta_i(s)$ is convex

4. By continuity of u_i one can show (a special case of Berge's maximum theorem) that β_i is upper hemi-continuous

5. The combined best-reply correspondence $\beta : S \rightrightarrows S$, defined by $\beta(s) = \times_{i \in I} \beta_i(s)$, inherits all these properties

6. All conditions in Kakutani's Fixed-Point Theorem are met!

4 Examples

- There are lots and lots of examples, in economics, biology, political economy etc., and most are not zero-sum and most involve more than two players

4.1 Cournot oligopoly competition

The market interaction:

1. Simultaneous decisions on output quantities $q_i \geq 0$, for firms $i = 1, 2, ..., n$. Write $Q = q_1 + q_2 + .. + q_n$.

2. Let $p = P(Q)$ be the (unique) market-clearing price when aggregate output is Q. Assume that P is continuous.

3. For each firm i let $C_i(q_i)$ be its cost for producing output quantity $q_i \geq 0$. Assume that C_i is continuous with $C_i(0) = 0$.

4. For each firm i, its profit is then given by

$$\pi_i(q_1, .., q_n) = P(Q)\, q_i - C_i(q_i)$$

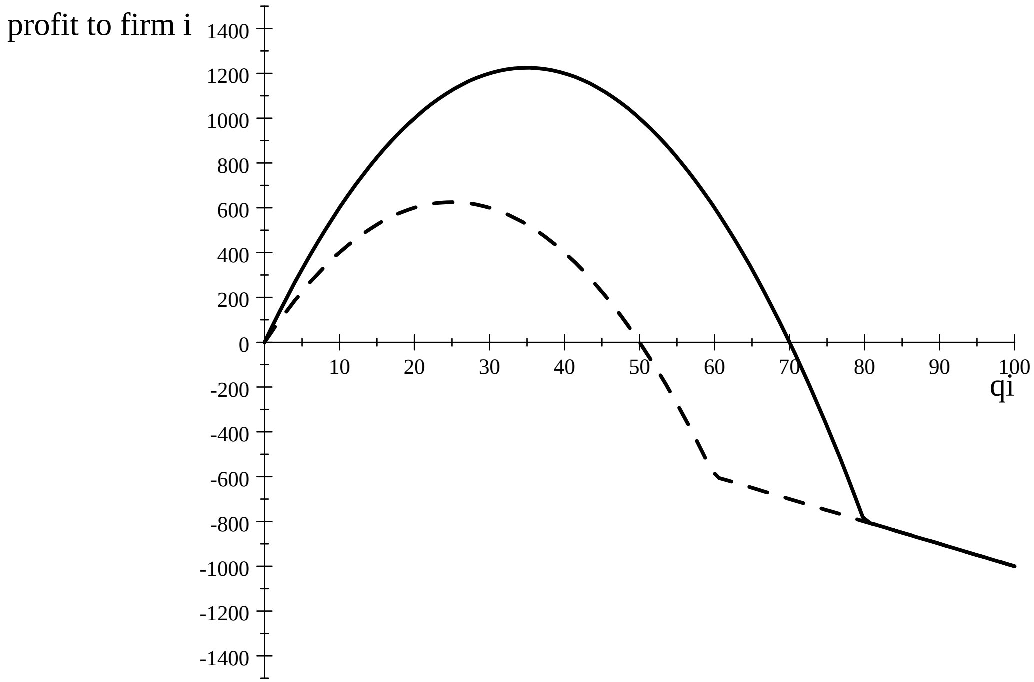
5. Assume that there for each firm i exists an output level $q_i^o > 0$ so high that $\pi_i(q_1, .., q_n) < 0$ for all $q_i > q_i^o$, irrespective of other firms' outputs

- We may represent any such market interaction as a normal-form game $G = \langle I, S, \pi \rangle$ with $S_i = \left[0, q_i^o\right]$ for all players i (since any firm may choose output level zero, and hence obtain profit zero)

- Check the list of conditions in Theorem 2.1!

- All conditions but one are met. Which one?

- Add the (reasonable) assumption that each firm's profit is quasi-concave in its own output level, for any given outputs of the other firms



4.2 Public goods

1. A community consisting of $n \geq 1$ individuals and a public good (a non-exclusive good that all consume, such as a clean kitchen, clean air, security against attacks.)

2. All individuals make contributions (or exerts an efforts) at some personal cost, and the sum of all contributions determines the amount of the public good. Let $x_i \geq 0$ denote the contribution of individual i. Assume that the utility of each individual i is

$$u_i(x_1, ..., x_n) = B_i\left(\sum_{j=1}^{n} x_j\right) - C_i(x_i)$$

for some continuous (benefit and cost) functions $B_i, C_i : \mathbb{R}_+ \to \mathbb{R}_+$, where

(a) all functions are continuous

(b) all benefit functions are concave and bounded

(c) all cost functions are convex and unbounded, with $C_i(0) = 0$

- Note that for each individual i there then exists an effort level $x_i^o > 0$ so high that $u_i(x_1, ..., x_n) < 0$ for all $x_i > x_i^o$, irrespective of others efforts.

- We may represent any such interaction as a normal-form game $G = \langle I, S, u \rangle$ with $S_i = \left[0, x_i^o\right]$ for all players i (since any individual may choose effort level zero, and hence obtain non-negative utility and for each individual i there exists an effort level $x_i^o > 0$)

- Check the list of conditions in Theorem 2.1!

5 Next lecture

- Finite games. Dominance relations, best replies, rationalizability and Nash equilibrium.

- Read ahead in the book!

6 A class-room experiment

- All present course participants i are invited to write an integer $s_i \in \{0, 1, 2, ..., 99, 100\}$, and your name, on a piece of paper

- All "bids" $s_1, ..., s_n$ will be collected

- We will calculate the average bid $\bar{s}$, and also three-quarters of this:

$$t = \frac{3}{4} \cdot \bar{s}$$

- I will pay 100 SEK to the participant who's bid is closest to this target t. If there are multiple such bids, these bidders will share the money