SF2972: Game theory

Mark Voorneveld, mark.voorneveld@hhs.se

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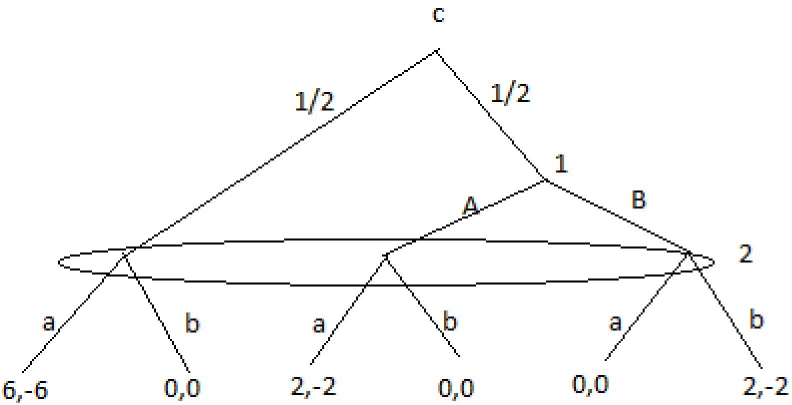
- Topic: extensive form games.
- Purpose: explicitly model situations in which players move sequentially; formulate appropriate equilibrium notions.
- Textbook (Peters): chapters 4, 5, 14. Reading guide towards end of each lecture's slides.

Defining games and strategies

Drawing a game tree is usually the most informative way to represent an extensive form game. Here is one with an initial (c)hance move:



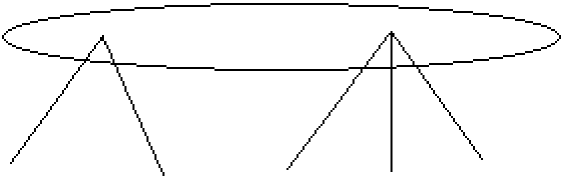
For LaTeX gurus: Is there a neat, quick way to draw game trees with TikZ?

Extensive form game: formal definition

- A (directed, rooted) tree; i.e. it has a well-defined initial node.
- Nodes can be of three types:
  1. chance nodes: where chance/nature chooses a branch according to a given/known probability distribution;
     Let τ assign to each chance node a prob distr over feasible branches.
  2. decision nodes: where a player chooses a branch;
  3. end nodes: where there are no more decisions to be made and each player i gets a payoff/utility given by a utility function u_i.
- A function P assigns to each decision node a player i in player set N who gets to decide there.
- Decision nodes $P^{-1}(i)$ of player i are partitioned into information sets.
  Nodes in an information set of player i are 'indistinguishable' to player i; this requires, for instance, the same actions in each decision node of the information set.
- If h is an information set of player i, write $P(h) = i$ and let $A(h)$ be the feasible actions in info set h.

Notational conventions

- p. 198: "Clearly, this formal notation is quite cumbersome and we try to avoid its use as much as possible. It is only needed to give precise definitions and proofs." *Draw tree!*
- Nodes in same information set: dotted lines between them (Peters' book) or enclosed in an oval (my drawings).
- Since nodes in an information set are indistinguishable, information sets like



are not allowed: since there are two branches in the left node and three in the right, they are easily distinguishable.

We call an extensive form game *finite* if it has finitely many nodes. An extensive form game has

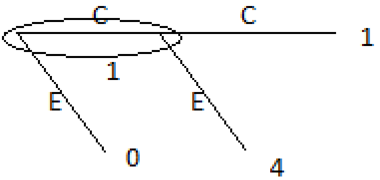
- *perfect information* if each information set consists of only one node.
- *perfect recall* if each player recalls exactly what he did in the past.
  Formally: on the path from the initial node to a decision node x of player i, list in chronological order which information sets of i were encountered and what i did there. Call this list the experience $X_i(x)$ of i in node x. The game has perfect recall if nodes in the same information set have the same experience.
- otherwise, the game has imperfect information/recall.

Convention: we often characterize nodes in the tree by describing the sequence of actions that leads to them. For instance:

- the initial node of the tree is denoted by $\emptyset$;
- node (a_1, a_2, a_3) is reached after three steps/branches/actions: first a_1, then a_2, then a_3.

Imperfect recall: absentminded driver

Two crossings on your way home. You need to (C)ontinue on the first, (E)xit on the second. But you don't recall *whether* you already passed a crossing.



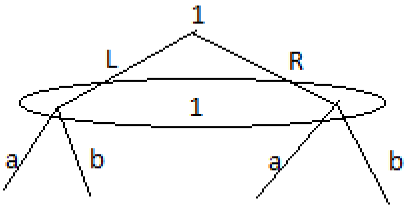
Only one information set, $\{\emptyset, C\}$, but with different experiences:

- in the first node: $X_1(\emptyset) = (\{\emptyset, C\})$
- in the second node:

  $$X_1(C) = (\underbrace{\{\emptyset, C\}}_{\text{1's first info set}}, \underbrace{C}_{\text{choice there}}, \underbrace{\{\emptyset, C\}}_{\text{resulting info set}})$$

- $X_1(\emptyset) \neq X_1(C)$: imperfect recall!

Player 1 forgets the initial choice:

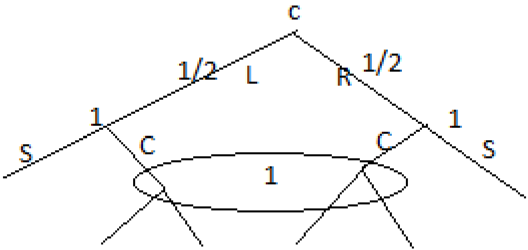


Different experiences in the two nodes of information set $\{L, R\}$:

- in the left node: $X_1(L) = (\underbrace{\emptyset}_{\text{initial node}}, \underbrace{L}_{\text{choice there}}, \underbrace{\{L, R\}}_{\text{resulting info set}})$
- in the right node: $X_1(R) = (\emptyset, R, \{L, R\})$.
- $X_1(L) \neq X_1(R)$: imperfect recall!

Third example of imperfect recall

Player 1 knew the chance move, but forgot it:



Different experiences in the two nodes of information set $\{(L, C), (R, C)\}$:

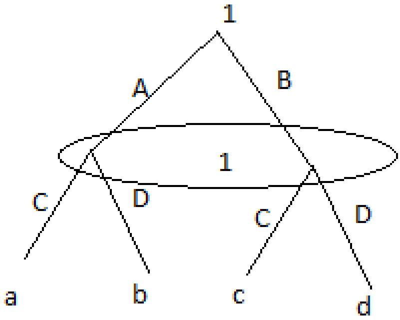
- in the left node:
  $X_1((L, C)) = (\underbrace{\{L\}}_{\text{1's first info set}}, \underbrace{C}_{\text{choice there}}, \underbrace{\{(L, C), (R, C)\}}_{\text{resulting info set}})$
- in the right node: $X_1((R, C)) = (\{R\}, C, \{(L, C), (R, C)\})$.
- $X_1((L, C)) \neq X_1((R, C))$: imperfect recall!

Pure, mixed, and behavioral strategies

- A *pure strategy* of player i is a function s_i that assigns to each information set h of player i a feasible action $s_i(h) \in A(h)$.
- A *mixed strategy* of player i is a probability distribution σ_i over i's pure strategies.
  $\sigma_i(s_i) \in [0,1]$ is the prob assigned to pure strategy s_i.
  'Global randomization' at the beginning of the game.
- A *behavioral strategy* of player i is a function b_i that assigns to each information set h of player i a probability distribution over the feasible actions $A(h)$.
  $b_i(h)(a)$ is the prob of action $a \in A(h)$.
  'Local randomization' as play proceeds.

Let us consider the difference between these three kinds of strategies in a few examples.

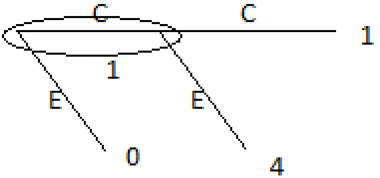
The difference between mixed and behavioral strategies



- Imperfect recall; 4 outcomes with payoffs a, b, c, and d.
- Four pure strategies, abbreviated AC, AD, BC, BD.
- Mixed strategies: probability distributions over the 4 pure strategies. A vector $(p_{AC}, p_{AD}, p_{BC}, p_{BD})$ of nonnegative numbers, adding up to one, with p_x the probability assigned to pure strategy $x \in \{AC, AD, BC, BD\}$.

- Behavioral strategies assign to each information set a probability distribution over the available actions. Since pl. 1 has 2 information sets, each with 2 actions, it is summarized by a pair $(p, q) \in [0,1] \times [0,1]$, where $p \in [0,1]$ is the probability assigned to action A in the initial node (and $1-p$ to B) and q is the probability assigned to action C in information set $\{A, B\}$ (and $1-q$ to D).
- Mixed strategy $(1/2, 0, 0, 1/2)$ assigns probability $1/2$ to each of the outcomes a and d. There is no such behavioral strategy:
  - reaching a with positive probability requires that $p, q > 0$;
  - reaching d with positive probability requires $p, q < 1$;
  - hence also b and c are reached with positive probability.

A trickier example: the absentminded driver revisited



- Pure strategies: C with payoff 1 and E with payoff 0.
- Mixed: let $p \in [0,1]$ be the prob of choosing pure strategy C and $1-p$ the prob of pure strategy E. Expected payoff: p.
- Behavioral: let $q \in [0,1]$ be the prob of choosing action C in the info set and $1-q$ the prob of choosing E in the info set. Expected payoff:

$$0 \cdot (1-q) + 4 \cdot q(1-q) + 1 \cdot q^2 = q(4-3q).$$

- No behavioral strategy is outcome-equivalent with $p = 1/2$ (why?)
- No mixed strategy is outcome-equivalent with $q = 1/2$ (why?)

Outcome-equivalence under perfect recall

Conclude: under imperfect recall, mixed and behavioral strategies might generate different probability distributions over end nodes.
Perfect recall helps to rule this out. We need a few definitions:
Each profile $b = (b_i)_{i \in N}$ of *behavioral strategies* induces an outcome $O(b)$, a probability distribution over end nodes.
How to compute $O(b)$ in finite games?
The probability of reaching end node $x = (a_1, \ldots, a_k)$, described by the sequence of actions/branches leading to it, is simply the product of the probabilities of each separate branch:

$$\prod_{\ell=0}^{k-1} b_{P(a_1,\ldots,a_\ell)}(a_1, \ldots, a_\ell)(a_{\ell+1}).$$

Likewise, each profile $\sigma = (\sigma_i)_{i \in N}$ of *mixed strategies* induces an outcome $O(\sigma)$, a probability distribution over end nodes.
How to compute $O(\sigma)$ in finite games?

- Let $x = (a_1, \ldots, a_k)$ be a node, described by the sequence of actions/branches in the game tree leading to it.
- Pure strategy s_i of player i is *consistent with* x if i chooses the actions described by x: for each initial segment $(a_1, \ldots, a_\ell)$ with $\ell < k$ and $P(a_1, \ldots, a_\ell) = i$:

$$s_i(a_1, \ldots, a_\ell) = a_{\ell+1}.$$

- The prob of i choosing a pure strategy s_i consistent with x is

$$\pi_i(x) = \sum \sigma_i(s_i),$$

  with summation over the s_i consistent with x.
- Similar for nature, whose behavior is given by function τ.
- The probability of reaching end node x is

$$\prod_{i \in N \cup \{c\}} \pi_i(x).$$

A mixed strategy σ_i and a behavioral strategy b_i of player i are *outcome-equivalent* if — given the pure strategies of the remaining players — they give rise to the same outcome:

$$\text{for all } s_{-i}: \quad O(\sigma_i, s_{-i}) = O(b_i, s_{-i}).$$

Theorem (Outcome equivalence under perfect recall)

In a finite extensive form game with perfect recall:

(a) *each behavioral strategy has an outcome-equivalent mixed strategy,*

(b) *each mixed strategy has an outcome-equivalent behavioral strategy.*

Proof sketch:

(a) Given beh. str. b_i, assign to pure strategy s_i the probability

$$\sigma_i(s_i) = \prod_h b_i(h)(s_i(h)),$$

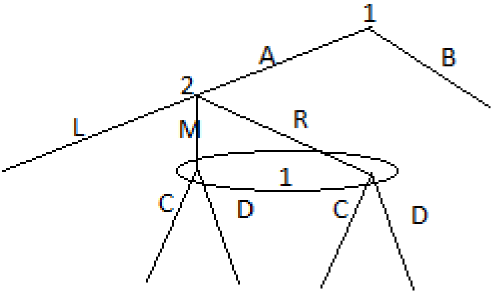
with the product taken over all info sets h of pl i.
Intuition: s_i selects action $s_i(h)$ in information set h. How likely is that?

(b) Given mixed str. σ_i. Consider an info set h of pl i and a feasible action $a \in A(h)$. How should we define $b_i(h)(a)$?
Consider any node x in info set h. The probability of choosing consistent with x is $\pi_i(x)$.
Perfect recall: $\pi_i(x) = \pi_i(y)$ for all $x, y \in h$.
Define

$$b_i(h)(a) = \frac{\pi_i(x, a)}{\pi_i(x)} \quad \text{if } \pi_i(x) > 0 \text{ (and arbitrarily otherwise)}$$

Intuition: conditional on earlier behavior that is consistent with reaching information set h, how likely is i to choose action a?

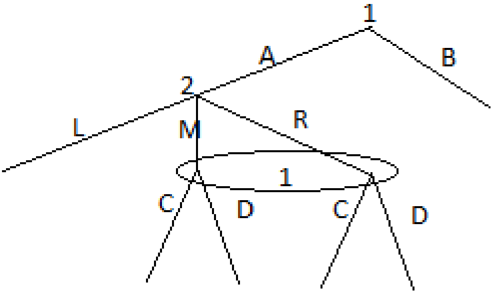
Example of outcome equivalent strategies



Question: Which behavioral strategy is outcome-equivalent with mixed strategy $(p_{AC}, p_{AD}, p_{BC}, p_{BD})$?
In 1's first information set, the prob that A is chosen is $p_{AC} + p_{AD}$.
In 1's second information set, the prob that C is chosen is computed as the probability of choosing C conditional on earlier behavior that is consistent with this information set being reached:

$$\frac{p_{AC}}{p_{AC} + p_{AD}}. \quad \text{(arbitrary if } p_{AC} + p_{AD} = 0\text{)}$$

Example of outcome equivalent strategies

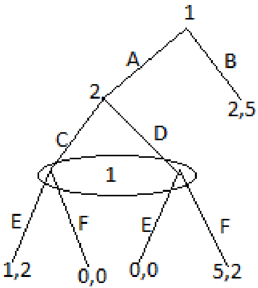


Question: Which mixed strategy is outcome equivalent with the behavioral strategy choosing A with prob p and C with prob q?

$$(p_{AC}, p_{AD}, p_{BC}, p_{BD}) = (pq, p(1-q), (1-p)q, (1-p)(1-q))$$

If $p = 0$, the 2nd info set is not reached: end node 'B' is reached with prob 1. Only pure strategies BC and BD are consistent with this node being reached. All mixed strategies with $p_{BC} + p_{BD} = 1$ are then outcome equivalent.

Homework exercise 1



(a) Show that the game above has perfect recall.

(b) For each mixed strategy σ_1 of player 1, find the outcome-equivalent behavioral strategies.

(c) For each behavioral strategy b_1 of player 1, find the outcome-equivalent mixed strategies.

Reading guide

1. definition extensive form games: slides 1–4, book §4.1, §14.1
2. examples (im)perfect recall: slides 5–7, book 45–46, 199
3. pure, mixed, behavioral strategies: slides 8–11, book 46–47, 199–200
4. outcome equivalence of mixed and behavioral strategies under perfect recall: slides 12–18, book 200–202

On the definition of strategies

For next lecture, think about the following: pure, mixed, and behavioral strategies specify what happens in *all* information sets of a player. Even in those information sets that cannot possibly be reached if those strategies are used. Why do you think that is the case?