SF2972: Game theory

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1. Signalling games
2. Bayesian games

Signalling games: examples

1. Michael Spence, 2001 Nobel Memorial Prize in Economics, job-market signalling model
   - A prospective employer can hire an applicant.
   - The applicant has high or low ability, but the employer doesn't know which.
   - Applicant can give a signal about ability, for instance via education.
2. Language, according to some evolutionary biologists, evolved as a way "to tell the other monkeys where the ripe fruit is." [Quote from Terry Pratchett: "It's very hard to talk quantum using a language originally designed to tell other monkeys where the ripe fruit is." Nightwatch]

Sometimes it makes sense to signal what your private information is, sometimes not.

Signalling games: model

1. Chance chooses a type t from some nonempty finite set T according to known prob distr $\mathbb{P}$ with $\mathbb{P}(t) > 0$ for all $t \in T$.
2. Pl. 1 (the sender) observes t and chooses a message $m \in M$ in some nonempty finite set of messages M.
3. Pl. 2 (the receiver) observes m (not t) and chooses an action $a \in A$ in some nonempty finite set of actions A.
4. The game ends with utilities $(u_1(t, m, a), u_2(t, m, a))$.

A pure strategy for player 1 is a function $s_1 : T \to M$ and a pure strategy for player 2 is a function $s_2 : M \to A$.

Separating and pooling equilibria in signalling games

In signalling games, it is common to restrict attention to equilibria (s_1, s_2, β), where

- s_1 and s_2 are pure strategies;
- assessment (s_1, s_2, β) is Bayesian consistent;
- assessment (s_1, s_2, β) is sequentially rational.

Sometimes it is in the sender's interest to try to communicate her type to the receiver by sending different messages for different types

$$s_1(t) \neq s_1(t') \quad \text{for all } t, t' \in T.$$

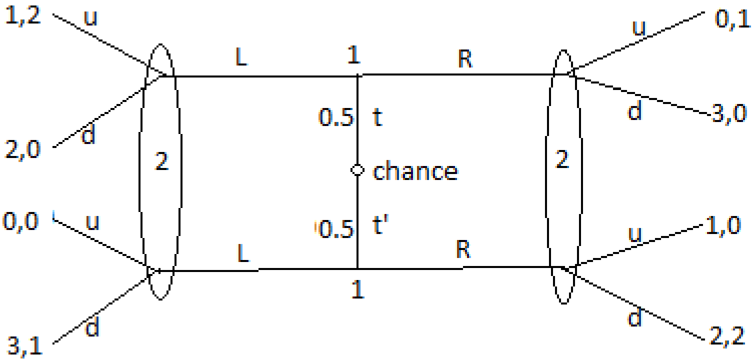
In such cases we call the equilibrium (s_1, s_2, β) a *separating equilibrium*.

In other cases, the sender might want to keep her signal a secret to the receiver and send the same message for each type:

$$s_1(t) = s_1(t') \quad \text{for all } t, t' \in T.$$

In such cases we call the equilibrium (s_1, s_2, β) a *pooling equilibrium*.

Signalling games: example



In the signalling game above:

(a) Find the corresponding strategic form game and its pure-strategy Nash equilibria.

(b) Determine (if any) the game's separating equilibria.

(c) Determine (if any) the game's pooling equilibria.

Answer (a):

- Pl. 1's pure strategies are pairs in $\{L, R\} \times \{L, R\}$, denoting the action after t and t', respectively.
- Pl. 2's pure strategies are pairs in $\{u, d\} \times \{u, d\}$, denoting the action after message L and R, respectively.
- Strategic form:

	(u, u)	(u, d)	(d, u)	(d, d)
(L, L)	$\frac{1}{2}, 1^*$	$\frac{1}{2}, 1^*$	$\frac{5}{2}^*, \frac{1}{2}$	$\frac{5}{2}, \frac{1}{2}$
(L, R)	$1^*, 1$	$\frac{3}{2}, 2^*$	$\frac{3}{2}, 0$	$2, 1$
(R, L)	$0, \frac{1}{2}$	$\frac{3}{2}, 0$	$\frac{3}{2}, 1^*$	$3^*, \frac{1}{2}$
(R, R)	$\frac{1}{2}, \frac{1}{2}$	$\frac{5}{2}^*, 1^*$	$\frac{1}{2}, \frac{1}{2}$	$\frac{5}{2}, 1^*$

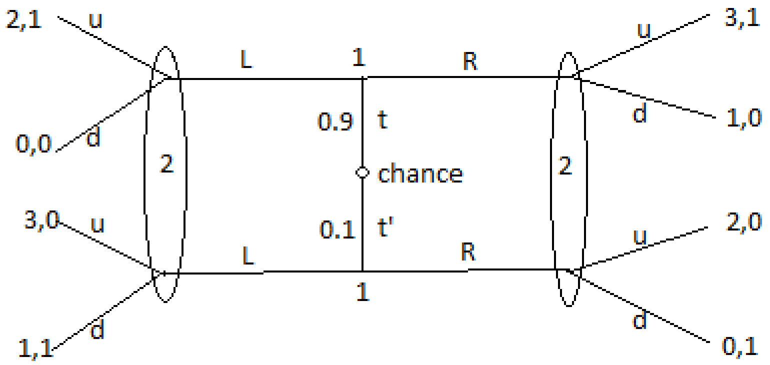
- Payoffs corresponding with best replies are starred, so there is a unique pure-strategy Nash equilibrium $((R, R), (u, d))$.

Answer (b): Separating equilibria must be Nash equilibria; but the only candidate $((R, R), (u, d))$ is of the pooling type: pl. 1 sends the same message R for both types. Conclude: no separating equilibria.

Answer (c):

- In (a), we found the candidate strategy profile $((R, R), (u, d))$.
- But what should the belief system be? Let $\alpha_1, \alpha_2 \in [0, 1]$ denote the prob assigned to the top node in the left and right info set, respectively.
- Bayesian consistency: requires that $\alpha_2 = \frac{1}{2}$, but imposes no constraints on α_1.
- Sequential rationality:
  1. Both info sets of pl. 1 and the right info set of pl. 2 are reached with positive prob. Since $((R, R), (u, d))$ is a NE, the players choose a best reply in those information sets.
  2. The left info set of pl. 2 is reached with zero prob. But the beliefs should be such that 2's action u is a best reply there.
  3. Pl. 2's payoff from u is $2\alpha_1 + 0(1 - \alpha_1)$ and from d is $0\alpha_1 + 1(1 - \alpha_1)$, so seq. rat. requires $\alpha_1 \geq \frac{1}{3}$.
- Conclude: Assessments (s_1, s_2, β) with strategies $(s_1, s_2) = ((R, R), (u, d))$ and belief system $\beta = (\alpha_1, \alpha_2) \in [1/3, 1] \times \{1/2\}$ are the game's pooling equilibria.

Homework exercise 4



In the signalling game above:

(a) Find the corresponding strategic form game and its pure-strategy Nash equilibria.

(b) Determine (if any) the game's pooling equilibria.

(c) Determine (if any) the game's separating equilibria.

Bayesian games are special imperfect information games where an initial chance move assigns to each player a privately known type. Knowing their own type, they choose an action (simultaneously, independently) and the game ends. Formally, the timing is as follows:

1. Chance chooses a vector $t = (t_i)_{i \in N}$ of types, one for each player, from a nonempty, finite product set $T = \times_{i \in N} T_i$ of types, according to known prob distr $\mathbb{P}$ with $\mathbb{P}(t) > 0$ for all $t = (t_i)_{i \in N} \in T$.
2. Each player i observes only her own type t_i and chooses an action a_i from some nonempty set A_i.
3. The game ends with utility $u_i(a_1, \ldots, a_n, t_1, \ldots, t_n)$ to player $i \in N = \{1, \ldots, n\}$.

Since $i \in N$ observes only $t_i \in T_i$, a pure strategy of player i is a function $s_i : T_i \to A_i$. Mixed and behavioral strategies are defined likewise.

Given her type, i updates her beliefs over other players' types t_{-i} using Bayes' Law: if she is of type t_i^*, she assigns probability

$$\mathbb{P}(t_{-i} \mid t_i^*) = \frac{\mathbb{P}(t_i^*, t_{-i})}{\mathbb{P}\{t \in T \mid t_i = t_i^*\}}$$

to the others having types $t_{-i} \in \times_{j \neq i} T_j$. Hence, her expected payoff given type t_i is

$$u_i(s_1, \ldots, s_n \mid t_i) = \sum_{t_{-i} \in T_{-i}} \mathbb{P}(t_{-i} \mid t_i) u_i(s_1(t_1), \ldots, s_n(t_n), t_1, \ldots, t_n).$$

It makes sense to require that each player i, for each possible type t_i, chooses her action optimally. That is, $s_i(t_i)$ should solve

$$\max_{a_i} \sum_{t_{-i} \in T_{-i}} \mathbb{P}(t_{-i} \mid t_i) u_i(s_1(t_1), \ldots, a_i, \ldots, s_n(t_n), t_1, \ldots, t_i, \ldots, t_n).$$

Strategies satisfying this requirement form a *Bayesian equilibrium* (in pure strategies; likewise for mixed and behavioral).

Question:

- Chance picks, with equal probability, game 1 or game 2:

game 1:

	L	R
T	$1, 1$	$0, 0$
B	$0, 0$	$0, 0$

game 2:

	L	R
T	$0, 0$	$0, 0$
B	$0, 0$	$2, 2$

- Player 1 learns which game was chosen, pl. 2 does not.
- Find all (pure-strategy) Bayesian equilibria.

Solution:

- Player 1 can be of two types, 1 or 2, depending on which game is chosen. Pl. 2 has only one type (omitted for convenience). Pl. 2 assigns equal probability to the two types of pl. 1.
- Pure strategy of player 1 is then a function $s_1 : \{1, 2\} \to \{T, B\}$, abbreviated as usual as a pair in $\{T, B\} \times \{T, B\}$.
- Pure strategy of player 2 (only one type) is simply an action from $\{L, R\}$.
- Distinguish two cases:

Case 1: Are there Bayesian equilibria where 2 chooses L?

- Best replies of 1 if her type is 1 (game 1 selected): action T.
- Best replies of 1 is type is 2: both T and B.
- Two candidates: $((T, T), L)$ and $((T, B), L))$.
- We made sure 1 plays a best reply to L, but does 2 choose a best reply?
- Pl. 2's expected payoffs against the strategies of pl. 1 are:

	L	R
(T, T)	$1/2^*$	0
(T, B)	$1/2$	1^*
(B, T)	0^*	0^*
(B, B)	0	1^*

  Payoffs corresponding to best replies are starred: L is a best reply to (T, T), but not to (T, B).
- Conclude: $((T, T), L)$ is a Bayesian equilibrium.

- Best replies of 1 if her type is 1: T and B.
- Best replies of 1 if her type is 2: B.
- Two candidates: $((T, B), R)$ and $((B, B), R)$.
- In the table above, we see that R is a best reply to (T, B) and to (B, B).
- Conclude: $((T, B), R)$ and $((B, B), R)$ are Bayesian equilibria.

Reading guide

1. Signalling games: slides 1–7, book §5.3 (skip 'intuitive criterion')
2. Bayesian games: slides 8–12, book §5.1, 5.2
3. You can find errata to Hans Peters' book on his homepage: `http://researchers-sbe.unimaas.nl/hanspeters/game-theory/`
4. On page 198, 4-th bullet, (1) should be "every path in T intersects h at most once".

Seminar Monday

Recall:

1. Send solutions to the four homework exercises in my lecture slides to my e-mail or hand them at the start of the tutorial on Monday.
2. Short solutions will be posted on the course web at a later time.