Short solutions precept 2; Mark Voorneveld

Exercise 1:

(a) The only nontrivial information set is $\{(A,C),(A,D)\}$ of player 1. In both nodes of this information set, player 1's experience is $(\emptyset, A, \{(A,C),(A,D)\})$. Since the experience is the same in all nodes of the information set, the game has perfect recall.

(b) Some definitions first:

1. A pure strategy of player 1 is a function that assigns to each information set of player 1 a feasible action. The four pure strategies can be summarized as $S_1 = \{A,B\} \times \{E,F\} = \{(A,E),(A,F),(B,E),(B,F)\}$.
2. A mixed strategy of player 1 is a probability distribution

$$(\sigma_1((A,E)), \sigma_1((A,F)), \sigma_1((B,E)), \sigma_1((B,F)))$$

over these pure strategies: $\sigma_1(s_1) \geq 0$ for all $s_1 \in S_1$ and $\sum_{s_1 \in S_1} \sigma_1(s_1) = 1$.

3. A behavioral strategy of player 1 is a function that assigns to each information set of player 1 a probability distribution over feasible actions. Here, it suffices to specify the probability $p = b_1(\emptyset)(A)$ that 1 assigns to A in the initial node (B has probability $1-p$) and the probability $q = b_1(\{(A,C),(A,D)\})(E)$ that 1 assigns to E in information set $\{(A,C),(A,D)\}$ (F has probability $1-q$).

Let σ_1 be a mixed strategy of player 1. Outcome-equivalent are the behavioral strategies $(p,q) \in [0,1] \times [0,1]$ with $p = \sigma_1((A,E)) + \sigma_1((A,F))$ and $q = \frac{\sigma_1((A,E))}{\sigma_1((A,E))+\sigma_1((A,F))}$ if the denominator is positive and $q \in [0,1]$ arbitrarily otherwise.

(c) Let $(p,q) \in [0,1] \times [0,1]$ be a behavioral strategy of player 1. Outcome-equivalent is the mixed strategy σ_1 with

$$(\sigma_1((A,E)), \sigma_1((A,F)), \sigma_1((B,E)), \sigma_1((B,F)) = (pq, p(1-q), (1-p)q, (1-p)(1-q)).$$

If $p = 0$, the second information set of pl. 1 is not reached: end node 'B' is reached with probability 1. Only pure strategies (B,E) and (B,F) are consistent with this node being reached and any mixed strategy σ_1 with $\sigma_1(B,E) + \sigma_1(B,F) = 1$ is outcome equivalent.

Exercise 2:

(a)

	C	D
(A,E)	$1,2^*$	$0,0$
(A,F)	$0,0$	$5^*,2^*$
(B,E)	$2^*,5^*$	$2,5^*$
(B,F)	$2^*,5^*$	$2,5^*$

(b) Above, payoffs corresponding with best replies are starred. So there are three pure Nash equilibria: $((A,F),D)$, $((B,E),C)$, and $((B,F),C))$.

(c) Consecutively eliminate:

1. (A,E): it is strictly dominated by (B,E) and (B,F);
2. C: it is weakly dominated by D;

3. (B,E) and (B,F): they are strictly dominated by (A,F).

The only pure strategy profile that survives this process is $((A,F),D)$.

(d) The game has two subgames: the entire game and a proper subgame starting at the decision node of player 2. The latter has strategic form

	C	D
E	$1,2$	$0,0$
F	$0,0$	$5,2$

and three Nash equilibria:

1. A pure-strategy equilibrium (E,C). If this is played in the proper subgame, then 1's payoff from A is 1 and from B is 2, so it is optimal to choose B. Conclude: one subgame perfect equilibrium is $((B,E),C)$. In behavioral strategies: 1 chooses B and E with probability one; 2 chooses C with probability one.
2. A pure-strategy equilibrium (F,D). If this is played in the proper subgame, then 1's payoff from A is 5 and from B is 2, so it is optimal to choose A. Conclude: one subgame perfect equilibrium is $((A,F),D)$. In behavioral strategies: 1 chooses A and F with probability one; 2 chooses D with probability one.
3. A mixed-strategy equilibrium where 1 chooses E with probability $1/2$ and 2 chooses C with probability $5/6$. If this is played in the proper subgame, then 1's payoff from A is $\frac{5}{6}$ and from B is 2, so it is optimal to choose B. Conclude: one subgame perfect equilibrium in behavioral strategies is: 1 chooses B with probability 1 and E with probability $\frac{1}{2}$; 2 chooses C with probability $\frac{5}{6}$.

Exercise 3: Denote an assessment by $(b_1, b_2, \beta) = ((p_A, p_E), p_C, \alpha)$. Here, $b_1 = (p_A, p_E) \in [0,1] \times [0,1]$ is 1's behavioral strategy specifying probabilities of choosing A and E in the relevant information sets; $b_2 = p_C \in [0,1]$ is 2's behavioral strategy specifying the probability of choosing C in his information set; belief system β is summarized by the probability $\alpha \in [0,1]$ it assigns to the left node (A,C) in 1's information set $\{(A,C),(A,D)\}$.

Recall: if (b_1, b_2, β) is a sequential equilibrium, (b_1, b_2) is subgame perfect. Using exercise 2(d), we find three sequential equilibria:

1. $(b_1, b_2, \beta) = ((p_A, p_E), p_C, \alpha) = ((0,1),1,1)$. To prove consistency, observe that this assessment is the limit of sequence
$$\left(\left(\frac{1}{n+1}, 1-\frac{1}{n+1}\right), 1-\frac{1}{n+1}, 1-\frac{1}{n+1}\right)_{n\in\mathbb{N}}$$
of completely mixed and Bayesian consistent assessments.

2. $(b_1, b_2, \beta) = ((p_A, p_E), p_C, \alpha) = ((1,0),0,0)$. To prove consistency, observe that this assessment is the limit of sequence
$$\left(\left(1-\frac{1}{n+1}, \frac{1}{n+1}\right), \frac{1}{n+1}, \frac{1}{n+1}\right)_{n\in\mathbb{N}}$$
of completely mixed and Bayesian consistent assessments.

3. $(b_1, b_2, \beta) = ((p_A, p_E), p_C, \alpha) = ((0,1/2),5/6,5/6)$. To prove consistency, observe that this assessment is the limit of sequence
$$\left(\left(\frac{1}{n+1}, \frac{1}{2}\right), \frac{5}{6}, \frac{5}{6}\right)_{n\in\mathbb{N}}$$
of completely mixed and Bayesian consistent assessments.

Exercise 4:

(a) Player 1 has pure strategy set $\{(L,L),(L,R),(R,L),(R,R)\}$, where the first letter indicates the action after signal t and the second letter the action after signal t'. Player 2 has pure strategy set $\{(u,u),(u,d),(d,u),(d,d)\}$, where the first letter indicates the action in the left information set (i.e., after player 1 chooses L) and the second letter the action in the right information set. The corresponding strategic form game is

	(u,u)	(u,d)	(d,u)	(d,d)
(L,L)	$\frac{21}{10},\frac{9}{10}$	$\frac{21}{10},\frac{9}{10}$	$\frac{1}{10},\frac{1}{10}$	$\frac{1}{10},\frac{1}{10}$
(L,R)	$2,\frac{9}{10}$	$\frac{18}{10},1$	$\frac{2}{10},0$	$0,\frac{1}{10}$
(R,L)	$3,\frac{9}{10}$	$\frac{12}{10},0$	$\frac{28}{10},1$	$1,\frac{1}{10}$
(R,R)	$\frac{29}{10},\frac{9}{10}$	$\frac{9}{10},\frac{1}{10}$	$\frac{29}{10},\frac{9}{10}$	$\frac{9}{10},\frac{1}{10}$

There are two pure-strategy Nash equilibria: $((L,L),(u,d))$ and $((R,R),(d,u))$.

(b) The equilibria in (a) are two candidates; but what restrictions do we need on the belief system? There are two nontrivial information sets; the belief system can be summarized by probability α_1 assigned to the top node in the left information set of player 2 and probability α_2 assigned to the top node in the right information set of player 2. Now consider the two candidate pooling equilibria separately:

1. In Nash equilibrium $((L,L),(u,d))$, Bayesian consistency:
   - requires $\alpha_1 = \frac{9}{10}$.
   - imposes no restriction on α_2, since the right information set of player 2 is reached with probability zero.

   Now consider sequential rationality:

   Both information sets of player 1 and the left information set of player 2 are reached with positive probability. Since $((L,L),(u,d))$ is a Nash equilibrium, the players choose a best reply in those information sets.

   Finally, the right information set is reached with probability zero, so beliefs there were not restricted by Bayesian consistency. But sequential rationality says that the beliefs do have to be such that player 2 chooses a best response in that information set. The expected payoffs to actions u and d, given the belief α_2, are

   $$1\cdot\alpha_2 + 0\cdot(1-\alpha_2) = \alpha_2 \quad \text{and} \quad 0\cdot\alpha_2 + 1\cdot(1-\alpha_2) = 1-\alpha_2,$$

   respectively. Action d is a best response provided $0 \le \alpha_2 \le \frac{1}{2}$.

   Conclude: assessments (s_1,s_2,β) with $(s_1,s_2) = ((L,L),(u,d))$ and belief system $\beta = (\alpha_1,\alpha_2) \in \left\{\frac{9}{10}\right\} \times \left[0,\frac{1}{2}\right]$ are pooling equilibria.
2. Similarly, assessments (s_1,s_2,β) with $(s_1,s_2) = ((R,R),(d,u))$ and belief system $\beta = (\alpha_1,\alpha_2) \in \left[\frac{1}{2},1\right] \times \left\{\frac{9}{10}\right\}$ are pooling equilibria.

(c) None, see (a).