Short solutions precept 2; Mark Voorneveld

Exercise 1:

(a) The only nontrivial information set is $\{(A,C),(A,D)\}$ of player 1. In both nodes of this information set, player 1's experience is $(\emptyset, A, \{(A,C),(A,D)\})$. Since the experience is the same in all nodes of the information set, the game has perfect recall.

(b) Some definitions first:

1. A pure strategy of player 1 is a function that assigns to each information set of player 1 a feasible action. The four pure strategies can be summarized as $S_1 = \{A,B\} \times \{E,F\} = \{(A,E),(A,F),(B,E),(B,F)\}$.
2. A mixed strategy of player 1 is a probability distribution

$$(\sigma_1((A,E)), \sigma_1((A,F)), \sigma_1((B,E)), \sigma_1((B,F)))$$

over these pure strategies: $\sigma_1(s_1) \geq 0$ for all $s_1 \in S_1$ and $\sum_{s_1 \in S_1} \sigma_1(s_1) = 1$.

3. A behavioral strategy of player 1 is a function that assigns to each information set of player 1 a probability distribution over feasible actions. Here, it suffices to specify the probability $p = b_1(\emptyset)(A)$ that 1 assigns to A in the initial node (B has probability $1-p$) and the probability $q = b_1(\{(A,C),(A,D)\})(E)$ that 1 assigns to E in information set $\{(A,C),(A,D)\}$ (F has probability $1-q$).

Let σ_1 be a mixed strategy of player 1. Outcome-equivalent are the behavioral strategies $(p,q) \in [0,1] \times [0,1]$ with $p = \sigma_1((A,E)) + \sigma_1((A,F))$ and $q = \frac{\sigma_1((A,E))}{\sigma_1((A,E))+\sigma_1((A,F))}$ if the denominator is positive and $q \in [0,1]$ arbitrarily otherwise.

(c) Let $(p,q) \in [0,1] \times [0,1]$ be a behavioral strategy of player 1. Outcome-equivalent is the mixed strategy σ_1 with

$$(\sigma_1((A,E)), \sigma_1((A,F)), \sigma_1((B,E)), \sigma_1((B,F)) = (pq, p(1-q), (1-p)q, (1-p)(1-q)).$$

If $p = 0$, the second information set of pl. 1 is not reached: end node 'B' is reached with probability 1. Only pure strategies (B,E) and (B,F) are consistent with this node being reached and any mixed strategy σ_1 with $\sigma_1(B,E) + \sigma_1(B,F) = 1$ is outcome equivalent.

Exercise 2:

(a)

	C	D
(A,E)	$1,2^*$	$0,0$
(A,F)	$0,0$	$5^*,2^*$
(B,E)	$2^*,5^*$	$2,5^*$
(B,F)	$2^*,5^*$	$2,5^*$

(b) Above, payoffs corresponding with best replies are starred. So there are three pure Nash equilibria: $((A,F),D)$, $((B,E),C)$, and $((B,F),C)$.

(c) Consecutively eliminate:

1. (A,E): it is strictly dominated by (B,E) and (B,F);
2. C: it is weakly dominated by D;

3. (B,E) and (B,F): they are strictly dominated by (A,F).

The only pure strategy profile that survives this process is $((A,F),D)$.

(d) The game has two subgames: the entire game and a proper subgame starting at the decision node of player 2. The latter has strategic form

	C	D
E	$1,2$	$0,0$
F	$0,0$	$5,2$

and three Nash equilibria:

1. A pure-strategy equilibrium (E,C). If this is played in the proper subgame, then 1's payoff from A is 1 and from B is 2, so it is optimal to choose B. Conclude: one subgame perfect equilibrium is $((B,E),C)$. In behavioral strategies: 1 chooses B and E with probability one; 2 chooses C with probability one.
2. A pure-strategy equilibrium (F,D). If this is played in the proper subgame, then 1's payoff from A is 5 and from B is 2, so it is optimal to choose A. Conclude: one subgame perfect equilibrium is $((A,F),D)$. In behavioral strategies: 1 chooses A and F with probability one; 2 chooses D with probability one.
3. A mixed-strategy equilibrium where 1 chooses E with probability $1/2$ and 2 chooses C with probability $5/6$. If this is played in the proper subgame, then 1's payoff from A is $\frac{5}{6}$ and from B is 2, so it is optimal to choose B. Conclude: one subgame perfect equilibrium in behavioral strategies is: 1 chooses B with probability 1 and E with probability $\frac{1}{2}$; 2 chooses C with probability $\frac{5}{6}$.

Exercise 3: Denote an assessment by $(b_1, b_2, \beta) = ((p_A, p_E), p_C, \alpha)$. Here, $b_1 = (p_A, p_E) \in [0,1] \times [0,1]$ is 1's behavioral strategy specifying probabilities of choosing A and E in the relevant information sets; $b_2 = p_C \in [0,1]$ is 2's behavioral strategy specifying the probability of choosing C in his information set; belief system β is summarized by the probability $\alpha \in [0,1]$ it assigns to the left node (A,C) in 1's information set $\{(A,C),(A,D)\}$.

Recall: if (b_1, b_2, β) is a sequential equilibrium, (b_1, b_2) is subgame perfect. Using exercise 2(d), we find three sequential equilibria:

1. $(b_1, b_2, \beta) = ((p_A, p_E), p_C, \alpha) = ((0,1),1,1)$. To prove consistency, observe that this assessment is the limit of sequence
$$\left(\left(\frac{1}{n+1}, 1-\frac{1}{n+1}\right), 1-\frac{1}{n+1}, 1-\frac{1}{n+1}\right)_{n\in\mathbb{N}}$$
of completely mixed and Bayesian consistent assessments.

2. $(b_1, b_2, \beta) = ((p_A, p_E), p_C, \alpha) = ((1,0),0,0)$. To prove consistency, observe that this assessment is the limit of sequence
$$\left(\left(1-\frac{1}{n+1}, \frac{1}{n+1}\right), \frac{1}{n+1}, \frac{1}{n+1}\right)_{n\in\mathbb{N}}$$
of completely mixed and Bayesian consistent assessments.

3. $(b_1, b_2, \beta) = ((p_A, p_E), p_C, \alpha) = ((0,1/2),5/6,5/6)$. To prove consistency, observe that this assessment is the limit of sequence
$$\left(\left(\frac{1}{n+1}, \frac{1}{2}\right), \frac{5}{6}, \frac{5}{6}\right)_{n\in\mathbb{N}}$$
of completely mixed and Bayesian consistent assessments.

Exercise 4:

(a) Player 1 has pure strategy set $\{(L,L),(L,R),(R,L),(R,R)\}$, where the first letter indicates the action after signal t and the second letter the action after signal t'. Player 2 has pure strategy set $\{(u,u),(u,d),(d,u),(d,d)\}$, where the first letter indicates the action in the left information set (i.e., after player 1 chooses L) and the second letter the action in the right information set. The corresponding strategic form game is

	(u,u)	(u,d)	(d,u)	(d,d)
(L,L)	$\frac{21}{10},\frac{9}{10}$	$\frac{21}{10},\frac{9}{10}$	$\frac{1}{10},\frac{1}{10}$	$\frac{1}{10},\frac{1}{10}$
(L,R)	$2,\frac{9}{10}$	$\frac{18}{10},1$	$\frac{2}{10},0$	$0,\frac{1}{10}$
(R,L)	$3,\frac{9}{10}$	$\frac{12}{10},0$	$\frac{28}{10},1$	$1,\frac{1}{10}$
(R,R)	$\frac{29}{10},\frac{9}{10}$	$\frac{9}{10},\frac{1}{10}$	$\frac{29}{10},\frac{9}{10}$	$\frac{9}{10},\frac{1}{10}$

There are two pure-strategy Nash equilibria: $((L,L),(u,d))$ and $((R,R),(d,u))$.

(b) The equilibria in (a) are two candidates; but what restrictions do we need on the belief system? There are two nontrivial information sets; the belief system can be summarized by probability α_1 assigned to the top node in the left information set of player 2 and probability α_2 assigned to the top node in the right information set of player 2. Now consider the two candidate pooling equilibria separately:

1. In Nash equilibrium $((L,L),(u,d))$, Bayesian consistency:
   - requires $\alpha_1 = \frac{9}{10}$.
   - imposes no restriction on α_2, since the right information set of player 2 is reached with probability zero.

   Now consider sequential rationality:

   Both information sets of player 1 and the left information set of player 2 are reached with positive probability. Since $((L,L),(u,d))$ is a Nash equilibrium, the players choose a best reply in those information sets.

   Finally, the right information set is reached with probability zero, so beliefs there were not restricted by Bayesian consistency. But sequential rationality says that the beliefs do have to be such that player 2 chooses a best response in that information set. The expected payoffs to actions u and d, given the belief α_2, are

   $$1\cdot\alpha_2 + 0\cdot(1-\alpha_2) = \alpha_2 \quad \text{and} \quad 0\cdot\alpha_2 + 1\cdot(1-\alpha_2) = 1-\alpha_2,$$

   respectively. Action d is a best response provided $0 \le \alpha_2 \le \frac{1}{2}$.

   Conclude: assessments (s_1,s_2,β) with $(s_1,s_2) = ((L,L),(u,d))$ and belief system $\beta = (\alpha_1,\alpha_2) \in \left\{\frac{9}{10}\right\} \times \left[0,\frac{1}{2}\right]$ are pooling equilibria.
2. Similarly, assessments (s_1,s_2,β) with $(s_1,s_2) = ((R,R),(d,u))$ and belief system $\beta = (\alpha_1,\alpha_2) \in \left[0,\frac{1}{2}\right] \times \left\{\frac{9}{10}\right\}$ are pooling equilibria.

(c) None, see (a).

Sequential equilibria in Selten's horse

Notation:

- $p \in [0,1]$ is the probability that player 1 chooses D;
- $q \in [0,1]$ is the probability that player 2 chooses d;
- $r \in [0,1]$ is the probability that player 3 chooses L;
- $\alpha \in [0,1]$ is the probability with which player 3 believes to be in the left node of 3's information set.

Sequential equilibria: all (p,q,r,α) with $p = q = 0$, $r \in [0,1/4]$, $\alpha = 1/3$ if $r \in (0,1/4]$ and $\alpha \in [0,1/3]$ if $r = 0$.

MOTIVATION: in 2's information set, d gives expected payoff $4r$, and c gives expected payoff 1. So distinguish 3 cases:

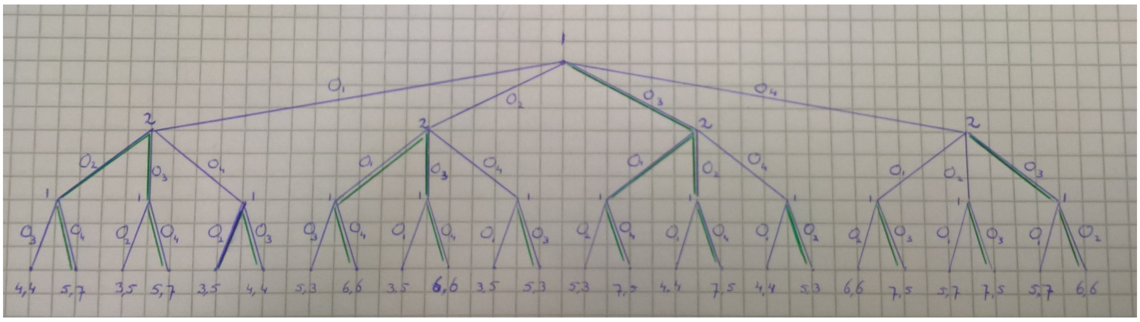
1. $r \in [0,1/4)$. By sequential rationality, 2 plays c and 1 plays C: $p = q = 0$. If $r \in (0,1/4)$, both of 3's actions must have the same expected payoff: $\alpha = 1/3$. If $r = 0$, R must have at least the same payoff as L: $\alpha \leq 1/3$.
2. $r = 1/4$. Both L and R must have the same expected payoff, so $\alpha = 1/3$. By sequential rationality, 1 plays C. Player 2, who is indifferent, must play c, otherwise the belief $\alpha = 1/3$ isn't (weakly) consistent. So again, $p = q = 0$.
3. $r \in (1/4,1]$. By sequential rationality, 2 plays d, 1 plays C. But 3's unique best response to (C,d) is to choose $r = 0$, contradicting the assumption that $r \in (1/4,1]$: no sequential equilibria in this case!

Verifying that the beliefs really are consistent is a standard limit argument.

Exercise 4.4(b) and (c) in Hans Peters (2015)

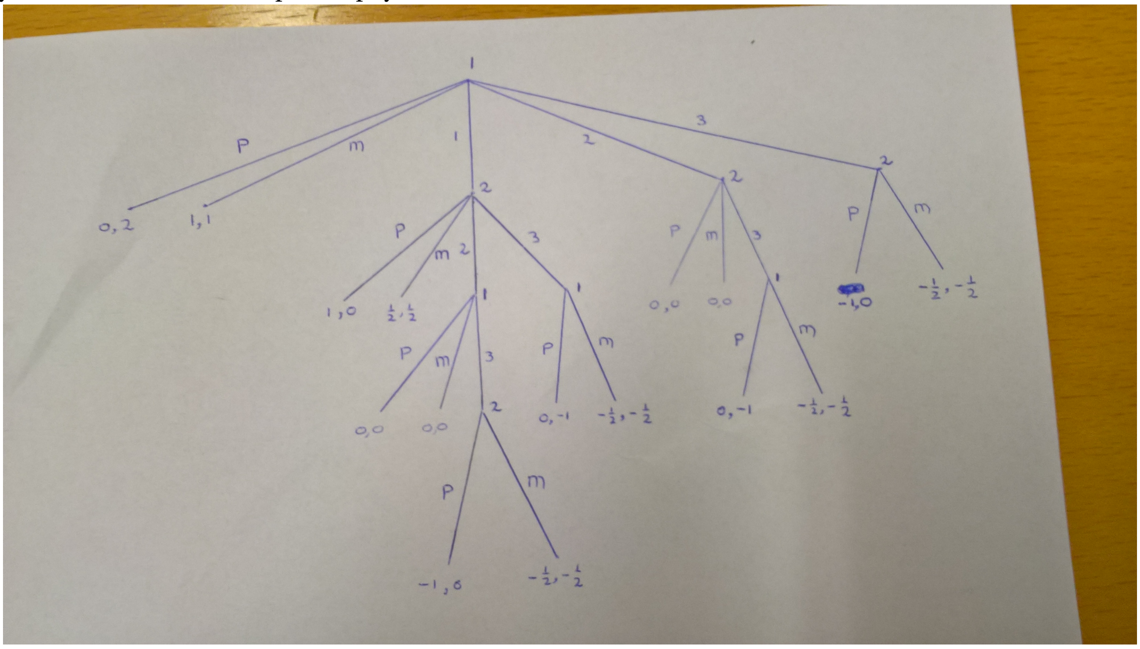
(b) Player 1 has 2^{14} pure strategies, player 2 has 3^4.

(c) I used backward induction to find the optimal branches in the game tree below; they are marked in green. Player 2 has 2 optimal actions in 3 of the subgames and we find $2^3 = 8$ subgame perfect Nash equilibria, all of which lead to player 1 receiving O_3 and O_4.



Exercise 4.5 in Hans Peters (2015)

(a) In the picture, the branches are labeled by P for pass, M for match, or a number 1, 2, 3 for a corresponding bid. At the end nodes, we give the expected payoffs. If a player gets the object, the payoff is the value (2) minus the bid, otherwise it is zero. Throughout the exercise we assume that the players want to maximize expected payoffs.



(b) Player 1 has $5 \cdot 3 \cdot 2 \cdot 2 = 60$ pure strategies, player 2 has $4 \cdot 3 \cdot 2 \cdot 2 = 48$ pure strategies.

(c) I find the subgame perfect Nash equilibria using backward induction. The shortest subgame arises if the current bid is 3, so that's where we start; we then proceed to longer subgames with current bids of 2, 1, and 0.

Our conclusion will be that there are four subgame perfect Nash equilibria in pure strategies and that they all result in the same outcome: 1 chooses M in the initial node.

If it is your turn in a subgame with current bid 3, you have 2 actions, pass P or match M.

1. P gives you payoff 0, your oppenent -1 (value 2 minus bid 3).
2. M gives you and your opponent expected payoff $-1/2$ (with probability $1/2$ you don't get the item; with probability $1/2$ you do, but then you pay 3 for an item of value 2).
3. So P is optimal, giving you payoff 0, the opponent payoff -1.

If it is your turn in a subgame with current bid 2, you have 3 actions, pass P, match M, or increase the bid to 3.

1. P gives you 0, your opponent 0.
2. M gives you 0, your opponent 0.
3. bid 3 gives a new subgame with current bid 3. By the previous step (but reversing roles, since it is your opponent's turn to move in the next subgame), your opponent will choose P giving you payoff -1 and your opponent payoff 0.
4. So both P and M are optimal, giving you and your opponent payoff 0.

If it is your turn in a subgame with current bid 1, you have 4 actions, pass P, match M, or increase the bid to 2 or 3.

1. P gives you 0 and your opponent 1.
2. M gives you and your opponent $1/2$.
3. bid 2 gives a new subgame with current bid 2, where you both get 0.
4. bid 3 gives a new subgame with current bid 3, where you get -1 and your opponent gets 0.
5. So M is optimal, giving you and your opponent payoff $1/2$.

If it is your turn in a subgame with current bid 0, you have 5 actions, pass P, match M, or increase the bid to 1, 2, or 3.

1. P gives you 0 and your opponent 2.
2. M gives you and your opponent 1.
3. bid 1 gives a new subgame with current bid 1, where you both get payoff $1/2$.
4. bid 2 gives a new subgame with current bid 2, where you both get payoff 0.
5. bid 3 gives a new subgame with current bid 3, where you get payoff -1 and your oppoent gets 0.
6. So M is optimal, giving you and your opponent payoff 1.

In subgames where the current bid was distinct from 2, the player whose turn it was to move has a unique optimal action: pass if the bid is high (3), match if the bid is low (0 or 1). In a subgame with current bid of 2, both match and pass are optimal. There are two subgames where the current bid is 2, so we find $2 \cdot 2 = 4$ subgame perfect equilibria. Note, however, that in all these equilibria, the game ends immediately with player 1 choosing M and expected payoff 1 to both players.

(d) A pure strategy profile is a Nash equilibrium if and only if player 1 chooses M in the initial node. This gives the same outcome, expected payoff 1 to both players, as in the subgame perfect Nash equilibria we found above.

MOTIVATION: Firstly, any strategy combination where 1 chooses M in the initial node is a Nash equilibrium:

1. Player 1 has no profitable deviation: strategies that keep choosing M in the initial node give the same outcome; so look at strategies that do anything other than M in the initial node. Choosing P gives payoff 0, which is worse than the payoff of 1 from choosing M. If 1 makes a bid $b \in \{1,2,3\}$, the player's payoff is 0 if she eventually doesn't get the item and at most $2 - b \leq 1$ if she does, so this isn't better than her payoff of 1 either.
2. Player 2 has no profitable deviation: since 1 chooses M, the game ends immediately and player 2's strategy is irrelevant for the outcome.

Conversely, in a Nash equilibrium, 1 must choose M in the initial node:

1. Strategies of player 1 starting with P or a bid of 2 or 3 are strictly dominated: since the object has value 2, the corresponding expected payoff from such strategies is at most 0, whereas M gives 1.
2. Strategies where 1 bids 1 in the initial node cannot be part of a Nash equilibrium either: player 2's unique best response would be to match the bid and get expected payoff $1/2$: passing or increasing the bid gives at most 0. But then player 1 also gets $1/2$, whereas deviating to M in the initial node would give a higher expected payoff of 1.