Short solutions problem set 3; Mark Voorneveld and Jörgen Weibull

Exercise 1:

(a) $(m_1, w_3), (m_2, w_1), (m_3, w_2)$, and w_4, w_5 remain single.

(b) $(m_1, w_3), (m_2, w_2), (m_3, w_1)$, and w_4, w_5 remain single.

(c) No. In any stable matching, w_4, w_5 remain single (rural hospital theorem), m_1 must match to w_3 (he does so both in the best and the worst match for men), leaving only the two matchings above.

Exercise 2:

(a) One iteration ends with stable match $(m_1, w_1), (m_2, w_2)$.

(b) One iteration ends with stable match $(m_1, w_2), (m_2, w_1)$.

(c) Letting ✓ denote an unacceptable candidate, the ranking matrix with w_1's stated preference is

	w_1	w_2
m_1	1, ✓	2, 1
m_2	2, 1	1, 2

If men propose, the stable match is $(m_1, w_2), (m_2, w_1)$. Looking at the original preferences, we see that both women get the man they like most. Under (a), they got the man they liked least.

Exercise 3: Following the hints in the exercise, consider the matching problem of exercise 2.

Since the two men and the two women find all partners of the opposite sex acceptable, there are two possible stable matchings: $\{(m_1, w_1), (m_2, w_2)\}$ and $\{(m_1, w_2), (m_2, w_1)\}$. Both were shown to be stable above.

So *if* a rule always picks a stable matching, it must pick one of those two in the matching problem of exercise 2. If it picks the former, let w_1 lie as in exercise 2(c). That problem has only one stable matching, $\{(m_1, w_2), (m_2, w_1)\}$. It is stable because it is the outcome of the DA algorithm; it is the only one since both men and women must be matched by the rural hospital theorem and w_1 only finds m_2 acceptable, leaving only one candidate! But this matching is better for w_1: w_1 has an incentive to lie about her preferences.

Analogously, if our rule were to pick the second matching, man m_1 could lie about his preferences and state that he only finds w_1 acceptable.

Exercise 4: Iteration 1 yields cycle $(2, h_5, 5, h_2, 2)$; iteration 2 yields cycles $(1, h_1, 1)$ and $(4, h_3, 3, h_4, 4)$. The resulting match is $(1, h_1), (2, h_5), (3, h_4), (4, h_3), (5, h_2)$.

Solution to Jörgen's exercise on repeated games:

(a) Each individual's utility function is continuous and strictly concave in the individual's own effort. Hence, each player has at most one best reply to each strategy profile. The partial derivative of $u_i(x_1,\dots,x_n)$ with respect to x_i is $g'(x_1+\cdots+x_n)/n - x_i$, so i's unique best reply must satisfy the first-order condition $x_i = g'(x_1+\cdots+x_n)/n$. Since this is the same equation for all partners, they will necessarily choose the same effort x^* in any Nash equilibrium.

This effort satisfies $nx^* = g'(nx^*)$. This equation has a unique solution: its left- and righthand side are continuous functions, the left is increasing, the right nonincreasing, so we find a unique intersection using the intermediate value theorem.

Aggregate effort $y^* = nx^*$ satisfies $y^* = g'(y^*)$. Thus, aggregate effort y^* is independent of n and $x^* = y^*/n$ is decreasing in n.

(b) If all partners exert the same effort x, then the sum of their utilities is $g(nx) - nx^2/2$. This is maximized iff x satisfies $g'(nx) = x$. As in (a), this equation has a unique solution $\hat{x}$. Comparing the condition $nx^* = g'(nx^*)$ in (a) with the condition $\hat{x} = g'(n\hat{x})$ in (b), it follows that $\hat{x} > x^*$ for all $n > 1$.

In (a), each partner has payoff $g(nx^*)/n - (x^*)^2/2$. Since $\hat{x}$ maximizes $x \mapsto g(nx)/n - x^2/2$, it follows that the partners are better off when they exert effort $\hat{x}$ for any $n > 1$.

(c) Let $\hat{u}$ (u^*) be a partner's period utility when they all play $\hat{x}$ (x^*). Then $\hat{u} > u^*$. Let u^+ be the maximal utility a partner i can obtain from unilaterally deviating when all others play $\hat{x}$. Then $u^+ = \max_{x_i} u_i(x_i, \hat{x}_{-i})$ and $u^+ > \hat{u}$. (The inequality is strict since it is not a NE for all to play $\hat{x}$ when $n > 1$) A strategy profile is a SPE iff there exists no profitable one-shot deviation. Evidently there exists no such deviation after any history in which some player has deviated. Hence, it suffices to study deviation incentives from the strategy path.

Following the strategies gives sequence of utilities $(\hat{u}, \hat{u}, \dots)$ with discounted utility $\hat{u}$. A one-shot deviation gives u^+ now, followed by a constant sequence of Nash equilibrium payoffs u^* and this sequence $(u^+, u^*, u^*, \dots)$ has discounted utility $(1-\delta)u^+ + \delta u^*$. So there exists no profitable one-shot deviation from that path iff $\hat{u} \geq (1-\delta)u^+ + \delta u^*$.

(d) In (a), $nx^* = g'(nx^*)$ becomes $x^* = 1/n$ with aggregate effort $y^* = 1$; in (b), $\hat{x} = g'(n\hat{x})$ becomes $\hat{x} = 1$ with aggregate effort $\hat{y} = n$; in (c), $\hat{u} = 1/2$, $u^* = (2n-1)/2n^2$, $u^+ = (2n^2-2n+1)/2n^2$, so the inequality for non-existence of a profitable one-shot deviation becomes $\delta \geq 1/2$.

(e) The minmax strategy against the other player is $x_i = 0$ and the other player's best reply to that is to choose effort $1/2$. Hence, each player's minimax level is $v^0 = 1/8$. Mutual minmaxing (i.e., both players choose effort 0) results in utility zero to both partners. By contrast, for $n = 2$, $\hat{u} = 1/2$ and $u^+ = 5/8$. Consider the mutual minmax strategy pair described in the proof of the Fudenberg-Maskin Folk theorem, with duration L periods. By the one-shot deviation principle, we need only consider one-shot deviations from the path of the strategy profile and from the first period in a punishment phase. The associated conditions for no profitable one-shot deviation are:

$$\hat{u} \geq (1-\delta)u^+ + (1-\delta)(\delta+\delta^2+\cdots+\delta^L)0 + \delta^{L+1}\hat{u},$$
$$(1-\delta)(1+\delta+\cdots+\delta^{L-1})0 + \delta^L\hat{u} \geq (1-\delta)v^0 + (1-\delta)(\delta+\delta^2+\cdots+\delta^L)0 + \delta^{L+1}\hat{u},$$

or, simplified:

$$(1-\delta^{L+1})\hat{u} \geq (1-\delta)u^+,$$
$$\delta^L\hat{u} \geq v^0.$$

Substituting $\hat{u} = \frac{1}{2}, u^{+} = \frac{5}{8}, v^{0} = \frac{1}{8}$, our question becomes *for which* $\delta \in (0,1)$ *can we find a suitable length* $L \in \mathbb{N}$ *of the punishment phase such that*

$$\frac{1-\delta^{L+1}}{1-\delta} \geq \frac{5}{4},$$
$$\delta^{L} \geq \frac{1}{4}.$$

The second inequality implies that δ must be at least $1/4$. To see that this condition is not only necessary, but also sufficient, note for $L = 1$, both inequalities simplify to $\delta \geq 1/4$.

Conclude: there is a sufficiently long punishment phase if and only if $\delta \geq 1/4$.