SF2972 GAME THEORY
Problem set 3

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Do homework exercises 1 to 4 in the lecture slides on matching, plus the following:

A partnership game: There are $n \geq 1$ partners who together own a firm. Each partner i chooses an effort level $x_i \geq 0$, resulting in total profit $g(y)$ for their firm, where $y = x_1 + \cdots + x_n$ is their aggregate effort. The profit function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous with $g(0) = 0$, and it is twice differentiable on $\mathbb{R}_{++}$ with $g' > 0$, and $g'' \leq 0$. The firm's profit is shared equally by the partners, and each partner's effort gives him or her (quadratic) disutility. The resulting utility level for each partner i is

$$u_i(x_1, \ldots, x_n) = \frac{1}{n} \cdot g\left(x_1 + \cdots + x_n\right) - x_i^2/2$$

where $(x_1, \ldots, x_n)$ is the effort profile.

(a) Suppose each partner i has to decide his or her effort x_i without observing the others' efforts. Show that the game has exactly one Nash equilibrium, and show that all partners make the same effort, x^*, in equilibrium. Is the individual equilibrium effort x^* increasing or decreasing in n, or is it independent of n? Is the aggregate equilibrium effort, $y^* = nx^*$, increasing or decreasing in n, or is it independent of n?

(b) Suppose that the partners can pre-commit to a common effort level, $x \geq 0$, the same for all. Let $\hat{x}$ be the common effort level that maximizes the sum of the partners' utilities. Characterize $\hat{x}$ in terms of an equation, and compare this level with the equilibrium effort x^* in (a), for $n = 1, 2, \ldots$. Are the partners better off now than in the equilibrium in (a)? How does this depend on n? Explain!

(c) Suppose that the interaction in (a) takes place every day, $t = 0, 1, 2, \ldots$ and suppose that all partners each day $t \geq 1$ can observe all partners' previous efforts. Moreover, assume that each partner strives to maximize the present value of her stream of daily utilities, discounted by the same factor $\delta \in (0, 1)$. The resulting utility level for each partner i is $(1 - \delta) \cdot \sum_{t=0}^{\infty} \delta^t u_i\left(x_1(t), \ldots, x_n(t)\right)$. For what range of discount factors $\delta \in (0, 1)$, if any, is it possible to induce, in subgame perfect equilibrium, each partner to each day exert the socially optimal effort level, $\hat{x}$ (see (b)), under the "threat" to punish any deviations by play forever of the daily Nash equilibrium in (a)? Write up the condition on δ as an inequality, and motivate it carefully and explain it!

(d) Now consider the special case of a linear profit function, $g(y) \equiv y$. Find explicit solutions for (a)-(c) and discuss how and why these solutions depend on $n \geq 1$, the number of partners in the firm.

(e) For the special case of a linear profit function, $g(y) \equiv y$, and with $n = 2$: For what range of discount factors $\delta \in (0,1)$ is it possible to induce, in subgame perfect equilibrium, each partner to each day exert the socially optimal effort level, $\hat{x}$ (see (b)), under the threat of (pure-strategy) mutual minmaxing, as in the Fudenberg-Maskin folk theorem for two-player games? Define precisely the behavior strategies that support such outcomes in this game. Compare the range of discount factor with that in (c) (for the linear profit function and $n = 2$).