Do homework exercises 1 to 4 in the lecture slides on matching, plus the following:

A PARTNERSHIP GAME: There are $n \geq 1$ partners who together own a firm. Each partner $i$ chooses an effort level $x_i \geq 0$, resulting in total profit $g(y)$ for their firm, where $y = x_1 + \cdots + x_n$ is their aggregate effort. The profit function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous with $g(0) = 0$, and it is twice differentiable on $\mathbb{R}_+$ with $g' > 0$, and $g'' \leq 0$. The firm’s profit is shared equally by the partners, and each partner’s effort gives him or her (quadratic) disutility. The resulting utility level for each partner $i$ is

$$u_i(x_1, \ldots, x_n) = \frac{1}{n} \cdot g(x_1 + \cdots + x_n) - \frac{x_i^2}{2}$$

where $(x_1, \ldots, x_n)$ is the effort profile.

(a) Suppose each partner $i$ has to decide his or her effort $x_i$ without observing the others’ efforts. Show that the game has exactly one Nash equilibrium, and show that all partners make the same effort, $x^*$, in equilibrium. Is the individual equilibrium effort $x^*$ increasing or decreasing in $n$, or is it independent of $n$? Is the aggregate equilibrium effort, $y^* = nx^*$, increasing or decreasing in $n$, or is it independent of $n$?

(b) Suppose that the partners can pre-commit to a common effort level, $x \geq 0$, the same for all. Let $\hat{x}$ be the common effort level that maximizes the sum of the partners’ utilities. Characterize $\hat{x}$ in terms of an equation, and compare this level with the equilibrium effort $x^*$ in (a), for $n = 1, 2, \ldots$. Are the partners better off now than in the equilibrium in (a)? How does this depend on $n$? Explain!

(c) Suppose that the interaction in (a) takes place every day, $t = 0, 1, 2, \ldots$ and suppose that all partners each day $t \geq 1$ can observe all partners’ previous efforts. Moreover, assume that each partner strives to maximize the present value of her stream of daily utilities, discounted by the same factor $\delta \in (0, 1)$. The resulting utility level for each partner $i$ is $(1 - \delta) \cdot \sum_{t=0}^{\infty} \delta^t u_i(x_1(t), \ldots, x_n(t))$. For what range of discount factors $\delta \in (0, 1)$, if any, is it possible to induce, in subgame perfect equilibrium, each partner to each day exert the socially optimal effort level, $\hat{x}$ (see (b)), under the “threat” to punish any deviations by play forever of the daily Nash equilibrium in (a)? Write up the condition on $\delta$ as an inequality, and motivate it carefully and explain it!
(d) Now consider the special case of a linear profit function, \( g(y) \equiv y \). Find explicit solutions for (a)-(c) and discuss how and why these solutions depend on \( n \geq 1 \), the number of partners in the firm.

(e) For the special case of a linear profit function, \( g(y) \equiv y \), and with \( n = 2 \):
For what range of discount factors \( \delta \in (0,1) \) is it possible to induce, in subgame perfect equilibrium, each partner to each day exert the socially optimal effort level, \( \hat{x} \) (see (b)), under the threat of (pure-strategy) mutual minmaxing, as in the Fudenberg-Maskin folk theorem for two-player games? Define precisely the behavior strategies that support such outcomes in this game. Compare the range of discount factor with that in (c) (for the linear profit function and \( n = 2 \)).