Repeated games

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1 Introduction

Q1: Can repetition enable "better" outcomes than "static" equilibrium?

- Peace instead of war?
- Resolution of the tragedy of the commons?
- Collusion in oligopolistic markets?
- Keeping together criminal gangs?

Q2: Can repetition enable "worse" outcomes than "static" equilibrium?

- Better for one party but worse for another? Worse for all parties?
Key concepts

Threats and promises

Punishments and rewards

Credibility

• Credible threats "cost nothing" but "credible promises" may be costly!
Example 1.1  Consider a repeated prisoners’ dilemma protocol (in monetary gains):

\[
\begin{array}{cc}
& c & d \\
\hline
\text{c} & 3,3 & 0,4 \\
\text{d} & 4,0 & 1,1 \\
\end{array}
\]

(a) Suppose this is played \( T = 100 \) times, each time as a simultaneous-move game, under perfect monitoring (of past moves), and that each player evaluates plays in terms of the sum of own monetary gains:

\[
\Pi_i = \sum_{t=1}^{T} \pi_i(a(t)) \quad i = 1, 2
\]

where \( a(t) \in \{c, d\}^2 \forall t \). If \( T = 100 \), how would you play? What does the extensive form look like? What is a strategy? Subgame? Find all SPE! Is cooperation possible in SPE?
(b) Suppose everything is as in (a), except that now $T$ is a geometrically distributed random variable. After each round, the game continues with probability $\delta \in (0, 1)$ to the next round, with statistically independent draws each time. Then

$$
\Pr [T = 1] = 1 - \delta, \quad \Pr [T = 2] = \delta (1 - \delta), \quad \Pr [T = 3] = \delta^2 (1 - \delta), \ldots
$$

(c) Suppose everything is as in (b), except that the random variable $T$ has a probability distribution with finite support, say $\Pr[T \leq 10^9] = 1$.

(d) Suppose everything as in (a),(b) or (c), except that now monitoring is imperfect. Two main cases: public monitoring (both players observe the same noisy signal about last round's play), private monitoring (each player observes a private noisy signal about last round’s play)
Example 1.2  Finitely repeated play of a coordination game with an added strictly dominated strategy:

\[
\begin{array}{ccc}
\text{a} & \text{b} & \text{c} \\
\text{a} & 3,3 & 0,0 & 8,0 \\
\text{b} & 0,0 & 1,1 & 0,0 \\
\text{c} & 0,8 & 0,0 & 7,7 \\
\end{array}
\]

Suppose each player adds up his or her period payoffs. Assume perfect monitoring.

Repeated play of \((b, b)\) gives payoff 1 to each player in each round. Can this be obtained in SPE?

Repeated play of \((a, a)\) gives payoff 3 to each player in each round. Can this be obtained in SPE?

Is it possible, in SPE, to obtain higher payoffs than \(3T\) for each player?
2 Infinitely repeated games with discounting

• Simultaneous-move stage game $G = \langle N, A, \pi \rangle$, for

$$N = \{1, \ldots, n\} \quad A = \times_{i=1}^{n} A_i \quad \pi : A \rightarrow \mathbb{R}^n$$

with each $A_i$ is finite (or, more generally, compact)

• Terminology: $a_i \in A_i$ “actions”

• Time periods $t = 0, 1, 2, \ldots$

• Perfect monitoring: all actions observed after each period

• Write $\alpha_i \in \Delta (A_i)$ if $\alpha_i$ is a randomized action choice, a "mixed action", by player $i$

• Write $\mathbb{N}$ for the non-negative integers (that is, including zero)
1. **Histories** $H = \bigcup_{t \in \mathbb{N}} H_t$

   In the initial period $t = 0$: $H_0 = \{h_0\}$ ($h_0$ is the “null history”)

   In any period $t > 0$: $h = \langle h_0, a(0), a(1), ..., a(t - 1) \rangle \in H_t = H_0 \times A^t$

2. **Plays**: infinite sequences of action profiles

   $$\tau = \langle a(0), a(1), ..., a(t), ... \rangle \in A^\infty$$

3. **Behavior strategies** $y_i : H \to \Delta(A_i)$

   (a) For any history $h \in H$: $y_i(h) = \alpha_i \in \Delta(A_i)$ is $i$’s (local) randomization, in the next period, over his or her action set

   (b) $Y_i$ denote the set of behavior strategies for player $i$, and let $Y = \times_{i \in \mathbb{N}} Y_i$
4. Each behavior-strategy profile \( y \in Y \), when used, recursively defines a play \( \tau \in A^\infty \):

(a) \( a(0) \in A \) is the realization of \( y(h_0) \in \square(A) \)

(b) \( a(1) \in A \) is the realization of \( y(h_0, a(0)) \in \square(A) \)

(c) \( a(2) \in A \) is the realization of \( y(h_0, a(0), a(1)) \in \square(A) \) etc.

5. Each player’s preferences over plays is assumed to be representable by the Bernoulli function

\[
v_i(\tau) = (1 - \delta) \sum_{t=0}^{\infty} \delta^t \pi_i[\alpha(t)]
\]

for some common discount factor \( \delta \in (0, 1) \)

This is the normalized present value of the stream of stage-game payoffs.
6. Payoff functions \( u_i : Y \rightarrow \mathbb{R} \) are defined as the *normalized expected present value* of the payoff stream:

\[
u_i(y) = (1 - \delta) \cdot \mathbb{E}_y \left[ \sum_{t=0}^{\infty} \delta^t \pi_i [a(t)] \right]
\]

This defines an *infinitely repeated game with discounting*, \( \Gamma^\delta \)

**Remark:** The assumption that preferences over plays take this simple additive form (over one’s own per-period payoffs) is a very strong assumption
3 Solution concepts

Definition 3.1 A behavior-strategy profile $y^*$ is a NE of $\Gamma^\delta$ if

$$u_i(y^*) \geq u_i(y_i, y_{-i}^*) \quad \forall i \in N, y_i \in Y_i$$

- Just as in the case of finite extensive-form games, a behavior-strategy profile is a NE if and only if it is sequentially rational on its own path.

- **Continuation strategies**: given any history $h \in H$, the restriction of a behavior-strategy profile $y$ to the subset of histories that begin with $h$:

  $$y|_h = (y_1|_h, \ldots, y_n|_h)$$
Recall that under perfect monitoring every history is the root of a subgame

Definition 3.2 A behavior-strategy profile \( y^* \) is a SPE of \( \Gamma^\delta \) if

\[
u_i \left( y^*_h \right) \geq u_i \left( y_i|h, y^*_{-i|h} \right) \quad \forall i \in N, y_i \in Y_i, h \in H
\]

Remark 3.1 Unconditional play of any NE of the stage game \( G \) in each period, can be supported in SPE in \( \Gamma^\delta \), for any \( \delta \) and for any time horizon \( T \leq +\infty \)

Remark 3.2 Unconditional play of any given sequence of NE of the stage game \( G \) can also be supported in SPE
4 The one-shot deviation principle

In dynamic programming: this principle is called unimprovability.

Definition 4.1 A one-shot deviation from a strategy $y_i \in Y_i$ is a strategy $y'_i \neq y_i$ that agrees with $y_i$ at all histories but one: $\exists! \ h^* \in H$ such that

$$y'_i(h) = y_i(h) \quad \forall h \neq h^*$$

Such a deviation from a strategy profile $y \in Y$ is profitable if

$$u_i|_{h^*} \left( y'_i, y_{-i} \right) > u_i|_{h^*}(y)$$
• Nash equilibria have no profitable one-shot deviations on their paths, but may have profitable one-shot deviations off their paths.

• But not so for subgame perfect equilibria:

**Proposition 4.1 (One-shot deviation principle)** A strategy profile \( y \) is a SPE of \( \Gamma^\delta \) if and only if \( \forall \) profitable one-shot deviation.

**Proof sketch:**

1. SPE \( \Rightarrow \) no profitable one-shot deviation

2. not SPE \( \Rightarrow \exists \) profitable one-shot deviation by “payoff continuity at infinity” (in class)
Example 4.1 Reconsider the Prisoners’ dilemma and use the one-shot deviation principle to test well-known strategy profiles for SPE, given some $\delta \in (0, 1)$: grim trigger, tit-for-tat, all D etc.
5 Folk theorems


Q: In infinitely repeated games with discounting and perfect monitoring, what payoff vectors (normalized expected present value of stream of stage-game payoffs) can be supported in SPE?

A: For sufficiently patient players (high \( \delta < 1 \)): any feasible and individually rational payoff vector in the stage game

- Why called "folk theorems"?

- Early versions: NE instead of SPE, limit average payoffs (no discounting) instead of present values under discounting
5.1 The Nash-threat folk theorem

- Suppose that each action set $A_i$ be compact (not necessarily finite), write $A = \times_{i \in N} A_i$ and let each stage-game payoff function $\pi_i : A \to \mathbb{R}$ be continuous.

- Then any payoff vector in the stage game that strictly Pareto dominates some stage-game NE can be supported in SPE if the players are sufficiently patient:

**Theorem 5.1 (Friedman, 1971)** Assume that $\nu = \pi (\hat{a}) > \pi (a^*)$ for some $\hat{a} \in A$ and some NE $a^* \in A$ in $G$. There exists a $\tilde{\delta} \in (0,1)$ such that $\nu$ is a SPE payoff outcome in $\Gamma^\delta$, for every $\delta \in [\tilde{\delta}, 1)$.

**Proof:** Let $y \in Y$ in $\Gamma^\delta$ be defined by $y(h_0) = \hat{a} \in A$, $y(h) = \hat{a}$ for all $h \in H$ in which all players took actions $\hat{a}$ in all preceding periods. For other $h \in H$: $y(h) = a^*$.
1. **On the path of \( y \):** No profitable one-shot deviation for player \( i \) iff

\[
(1 - \delta) \cdot M_i + \delta \cdot \pi_i (a^*) \leq \pi_i (\hat{a})
\]

(1)

where \( M_i = \max_{a_i \in A_i} \pi_i (a_i, \hat{a}_{-i}) \) (and note that \( M_i \geq \pi_i (\hat{a}) > \pi_i (a^*) \))

(a) Inequality (1) holds iff

\[
\delta \geq \delta_i = \frac{M_i - \pi_i (\hat{a})}{M_i - \pi_i (a^*)}
\]

(b) Let \( \bar{\delta} = \max_{i \in N} \delta_i \). Then \( \bar{\delta} < 1 \).

2. **Off the path of \( y \):** the stage-game NE \( a^* \) is prescribed in each period after any such history \( h \), so there is no profitable one-shot deviation
5.2 Example: Cournot duopoly

• Two identical firms, producing the same good, for which the demand function is

\[ D(p) = 100 - p \]

in each time period \( t = 1, 2, \ldots \)

• No fixed costs and a constant marginal production cost of \( c \geq 0 \) per unit

• Each firm \( i \) independently decides on its output, \( q_i(t) \), in each period \( t = 0, 1, 2, \ldots \)

• The resulting market price in period \( t \):

\[ p(t) = 100 - [q_1(t) + q_2(t)] \]
• Profits in period $t$:

$$\pi_i [q(t)] = (100 - [q_1(t) + q_2(t)] - c) \cdot q_i(t)$$

• Perfect monitoring: past outputs are observed (or, equivalently, past prices are observed)

• The stage game $G$ has a unique NE:

$$q_1 = q_2 = q^* = \frac{100 - c}{3}$$

• Let $Q^* = 2q^*$. This industry output exceeds monopoly industry output $\hat{Q}$:

$$\hat{Q} = \frac{1}{2}(100 - c) < \frac{2}{3}(100 - c) = Q^*$$
• Equilibrium industry profit fall short of monopoly industry profit:

\[ \Pi^* = 2 \left( \frac{100 - c}{3} \right)^2 < \left( \frac{100 - c}{2} \right)^2 = \widehat{\Pi} \]

• Note that the sum of profits is a function of the sum of outputs:

\[ \pi_1 + \pi_2 = (100 - (q_1 + q_2) - c) \cdot (q_1 + q_2) \]
• Suppose infinitely repeated with discount factor $\delta$ (for example $\delta = e^{-r\Delta}$ where $r$ is the interest rate and $\Delta$ the period length)

• Consider the following pure (behavior) strategy, $s_i^*$: start out with some quantity $\hat{q}_i \in (0, 100)$, and supply this output in all future periods, as long as no deviation from these output levels, $\hat{q} = (\hat{q}_1, \hat{q}_2)$ has been observed. If a deviation occurs: play the static Cournot equilibrium, $q^*$, in all future periods.

• No profitable one-shot deviations in any history containing a deviation from $\hat{q}$. The strategy pair $(s_1^*, s_2^*)$ is thus a SPE iff

$$\pi_i (\hat{q}) \geq (1 - \delta) \cdot \max_{q_i \in [0, 100]} \pi_i (q_i, \hat{q}_{-i}) + \delta \cdot \pi_i (q^*) \quad \text{for } i = 1, 2$$
• Possible to support also other outcomes in SPE? Lower than static Cournot profits for one firm, or even for both firms?
6 General folk theorems

Definition 6.1 An action profile $a = (a_1, \ldots, a_n) \in A$ is a minmax action-profile against player $i$ if

$$a_{-i} \in A_{-i}^0 = \arg \min_{a_{-i}} \left( \max_{a_i} \pi_i (a_i, a_{-i}) \right)$$

- It is as if the others gang up to jointly punish $i$ and $i$, knowing their punishment $(a_{-i})$ defends her/himself as best she/he can.

Definition 6.2 Player $i$’s minmax value:

$$v_i^0 = \min_{a_{-i}} \left( \max_{a_i} \pi_i (a_i, a_{-i}) \right)$$

Definition 6.3 A payoff vector $v \in \mathbb{R}^n$ is strictly individually rational if $v > v^0$. 
• In some games the resulting minmax value can be (much) lower if the punishers use mixed strategies

• Reconsider the Prisoner’s dilemma, the matching-pennies game, a 2x2 coordination game

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• What are the minmax vectors under pure/mixed minmaxing?
Definition 6.4  The set of feasible payoff vectors in the stage game $G$ is the convex hull of the direct payoff image of the action space:

$$V = co [\pi (A)] \subset \mathbb{R}^n$$

- Why is convexification natural?

Definition 6.5  The set of feasible and strictly individually rational payoff vectors in the stage game $G$:

$$V^* = \{ v \in V : v > v^0 \}$$

- Reconsider the above examples!
6.1 Two-player games

- Assume \( n = 2, \ A = A_1 \times A_2 \) compact and \( \pi : A \to \mathbb{R}^2 \) continuous

**Definition 6.6** A mutual minmax profile in \( G \) is an action profile \( (a_1^0, a_2^0) \in A \) such that \( a_1^0 \in A_1 \) is a minmax action against 2 and \( a_2^0 \in A_2 \) a minmax action against 1.

- Note that \( \pi (a_1^0, a_2^0) \leq v^0 \) (since a player’s minmax action is not necessarily a best-reply to the other’s minmax action)
Main result: Any payoff vector in the stage game that strictly Pareto dominates the minmax payoff vector can be supported in SPE if the players are sufficiently patient. Proof: Threat of temporary mutual minmaxing.

Theorem 6.1 (Fudenberg and Maskin, 1986) Let $n = 2$, and suppose $\hat{a} \in A$ is such that $\pi(\hat{a}) > v^0$. There exists a $\bar{\delta} \in (0, 1)$ such that play of $\hat{a} \in A$ in each period is supported by a SPE in $\Gamma^\delta$, for any $\delta \in [\bar{\delta}, 1)$. 
Proof sketch:

Given \( \hat{a} \in A \) such that \( \pi(\hat{a}) > v^0 \), consider a behavior-strategy profile \( y = (y_1, y_2) \) in the repeated game, with "penalty duration" \( L \):

1. Start by playing \( \hat{a} = (\hat{a}_1, \hat{a}_2) \), and play \( \hat{a} \) if \( \hat{a} \) was always played so far.

2. Also play \( \hat{a} \) if sometime in the past the mutual minmax profile \( a^0 \) was played for \( L \) consecutive periods after which no other action pair than \( \hat{a} \) was ever played.

3. For all other histories: play \( a^0 \)
• $L$ has to be long enough to deter deviations in phases 1 and 2, but short enough to deter deviation in phase 3. Such an $L$ always exists!

• Use the one-shot deviation principle!

  – One-shot deviations in phases 1&2 unprofitable iff
    \[
    \max_{a_i \in A_i} \pi_i (a_i, \hat{a}_-i) - \pi_i (\hat{a}) < \left( \delta + \delta^2 + \ldots + \delta^L \right) \left[ \pi_i (\hat{a}) - \pi_i (a^0) \right]
    \]

  – One-shot deviations in phase 3 unprofitable iff
    \[
    v^0_i - \pi_i (a^0) \leq \delta^L \cdot \left[ \pi_i (\hat{a}) - \pi_i (a^0) \right]
    \]

• Draw picture in class
• Can this theorem explain why two rational persons stand in a street beating each other with a stick?

• Reconsider the Cournot duopoly example!
6.2 Games with more than two players

- For $n > 2$ there may exist no mutual minmax action-profile:

\[
\begin{array}{c|cc}
L & R & \\
\hline
U & 1,1,1 & 0,0,0 \\
D & 0,0,0 & 0,0,0 \\
\end{array}
\quad
\begin{array}{c|cc}
L & R & \\
\hline
U & 0,0,0 & 0,0,0 \\
D & 0,0,0 & 1,1,1 \\
\end{array}
\]

- A player can unilaterally deviate from minmaxing of another player, and obtain a payoff 1, instead of the minmax value 0.

- The proof for $n = 2$ cannot be generalized. Not only that, the claim is not valid for generally valid for $n > 2$!

**Definition 6.7** Two players in $G$, say $i$ and $j$, have **equivalent** payoff functions if $\pi_j = \alpha \pi_i + \beta$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$. 
• The so-called NEU condition, or Non-Equivalent-Utilities condition: no pair of players have equivalent payoffs functions

• Assume that $A = \times_{i=1}^{n} A_i$ is compact and $\pi : A \to \mathbb{R}^n$ is continuous

**Theorem 6.2 (Abreu, Dutta and Smith, 1994)** Assume $G$ satisfies NEU. Suppose $\hat{a} \in A$ is such that $\pi(\hat{a}) > v^0$. Then there exists a $\bar{\delta} \in (0, 1)$ such that play of $\hat{a} \in A$ in each period is supported by a SPE in $\Gamma^\delta$, for every $\delta \in [\bar{\delta}, 1)$.

• See Abreu, Dutta and Smith (1994) and/or Mailath & Samuelson (2006)
7 Concluding comment

• Note the *neutrality* of the folk theorems: they do *not* say that repetition will necessarily lead to cooperation, only that it *enables* cooperation *if* players are sufficiently patient.

• Interesting implications of the folk theorem also for "bad" outcomes.