SF2972 GAME THEORY
Solutions to Problem set 1

Jorgen Weibull

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1. Peters (2015) Problem 2.5 (Problem 2.3 in the first edition). **Solution: The normal form of this game is**

	R	S	P
R	$0,0$	$1,-1$	$-1,1$
S	$-1,1$	$0,0$	$1,-1$
P	$1,-1$	$-1,1$	$0,0$

It has a unique NE: each party randomizes uniformly, $(1/3, 1/3, 1/3)$

2. Peters (2015) Problem 3.3. **Solution: (a) Strategy W strictly dominated strategy Z.**

	W	X	Y	Z
T	$6,6$	$4,4$	$1,2$	$8,5$
B	$4,5$	$6,6$	$2,8$	$4,4$

(b) **A mixed strategy** $p \cdot W + (1-p)Y$ **strictly dominates strategy X iff** $6p + 2(1-p) > 4$ and $5p + 8(1-p) > 6$. **Both inequalities hold iff** $1/2 < p < 2/3$. **(c) NE are** (T, W), (B, Y) **and the mixed profile** $x_1 = (3/7, 4/7)$ **and** $x_2 = (1/3, 0, 2/3, 0)$.

3. Peters (2015) Problem 3.4. **Solution: See book**.

4. Consider the 2-player normal-form game:

	A	B
A	a,b	$0,0$
B	$0,0$	$1,1$

for arbitrary payoffs $a, b > 0$.

   (a) Draw a diagram showing each player's set of best replies to any mixed-strategy profile. [Suggestion: let the horizontal axis be 1's probability (x) for playing her first pure strategy, and let the vertical axis be 2's probability (y) for playing his first pure strategy.]

Solution: $y^* = 1/(1+a)$ and $x^* = 1/(1+b)$ are horizontal and vertical lines of indifference of each player.

(b) Use the diagram to identify the set of Nash equilibria. **Solution: two strict and one mixed.**

(c) Study how the mixed Nash equilibrium depends on a and b, in particular if 1's probability depends on her payoff a or on the other player's payoff b. Explain! **Solution: (x^*, y^*) depends on the other player's payoff.**

(d) Calculate the (expected) payoffs in the mixed equilibrium and compare with the payoffs in the two strict equilibria. **Solution**: $\pi_1^* = a/(1+a)$ **and $\pi_2^* = b/(1+b)$. Less than both strict NE payoffs.**

(e) Show that each player's mixed Nash equilibrium strategy also is the player's min-max strategy against the other player. **Solution:**

$$\min_x \max_y \pi_2(x,y) = \min_x \max_y [bxy + (1-x)(1-y)]$$

where

$$\max_y [bxy + (1-x)(1-y)] = \begin{cases} 1-x & \text{if } x < x^* \\ b/(1+b) & \text{if } x \leq x^* \\ bx & \text{if } x > x^* \end{cases}$$

Hence, 1's maxmin strategy against 2 is precisely x^*

5. Consider a two-player simultaneous-move game G with normal form

	L	M	R
A	$8, 11$	$-3, 0$	$0, 0$
B	$9, -1$	$4, 1$	$0, 0$
C	$0, -2$	$0, 0$	$1, 4$

(a) Find all *rationalizable* pure strategies in G. **Solution: In a two-player game, a pure strategy is rationalizable if and only if it is not iteratively strictly dominated. Strategy A is strictly dominated (by a mixture of B and C). Once A has been eliminated, L becomes strictly dominated (by M, and by R). No more strategies can be eliminated, so the answer is: B, C, M, and R.**

(b) Find all Nash equilibria (in pure and mixed strategies). **Solution: All NE (in all games) have support among the rationalizable strategies. Hence we can focus on the block game on $T = T_1 \times T_2$ with $T_1 = \{B, C\}$ and $T_2 = \{M,R\}$. There are two pure and strict NE: (B, M) and (C, R). There is one mixed NE, x^*, in which player 1 plays B with probability $4/5$ and C with probability $1/5$ (to keep 2 indifferent), and player 2 plays M with probability $1/5$ and R with probability $4/5$. There are no other NE.**

(c) Find all (normal-form) *perfect equilibria* (in pure and mixed strategies). **Solution: Being strict, both (B, M) and (C, R) are perfect. All perfect equilibria (in all games) are undominated, and the converse is true for all two-player games. Since none of the strategies B, C, M, R is weakly dominated, also x^* is perfect.**

(d) Find all *proper equilibria* (in pure or mixed strategies). **Solution: Being strict, both (B, M) and (C, R) are proper. In order to see if the mixed NE is proper, we need to find a sequence of ε-proper strategy profiles converging to it. Let $x_1^\varepsilon = \frac{1}{5+\varepsilon}(\varepsilon, 4, 1)$ and $x_2^\varepsilon = \frac{1}{5+10\varepsilon}(\varepsilon, 1, 4 + 9\varepsilon)$. Then x^ε is ε-proper and $x^\varepsilon \to x^*$ as $\varepsilon \to 0$, so x^* is proper.**

6. Consider the following normal-form game G, arising from price competition between two firms with the same average and marginal cost $c = 1$ per unit facing aggregate demand $D(p) = 10 - 2p$. (Hence, the monopoly price in this market is $p = 3$ and the monopoly profit is 8.) Each firm i is constrained to choose an integer price, $p_i \in P = \{0, 1, 2, 3, 4\}$. In the following payoff bi-matrix, firm 1 chooses row and firm two column:

$p_1 \backslash p_2$	0	1	2	3	4
0	$-5,-5$	$-10,0$	$-10,0$	$-10,0$	$-10,0$
1	$0,-10$	$0,0$	$0,0$	$0,0$	$0,0$
2	$0,-10$	$0,0$	$3,3$	$6,0$	$6,0$
3	$0,-10$	$0,0$	$0,6$	$4,4$	$8,0$
4	$0,-10$	$0,0$	$0,6$	$0,8$	$3,3$

(a) Find all strictly dominated pure strategies. **Solution: First note that the game is symmetric, so if some strategy h for player 1 is strictly or weakly dominated, then this is also true for the same strategy for player 2. Clearly strategy 0 is strictly dominated by any one of the other pure strategies. No other pure strategy is strictly dominated.**

(b) Find all pure-strategy Nash equilibria. **Solution: There are 2 such equilibria:** $(1, 1)$ **and** $(2, 2)$.

(c) Find all weakly dominated strategies. **Solution: Strategies** 0, 1 **and** 4 **are weakly dominated**.

(d) Find all perfect pure-strategy Nash equilibria. **Solution: The only undominated NE is** $(2, 2)$**. Hence perfect.**

7. Lecture Notes Example 12. **Solution: See LN.**

8. The example at the end of Sethi and Weibull (2016). **Solution: The normal form of this game, with all payoffs multiplied by 5, is:**

	bid 0	bid 1	bid 2
bid 0	$5, 5$	$0, 6$	$0, 2$
bid 1	$6, 0$	$1, 1$	$-4, 2$
bid 2	$2, 0$	$2, -4$	$-3, -3$

It is easily verified that there is no NE in pure strategies, and that each player is indifferent between all his or her pure strategies if and only if the other player randomizes $(1/5, 3/5, 1/5)$.