KTH Teknikvetenskap

SF2972 Game Theory
Written Exam with Solutions
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Part A – Classical Game Theory

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1. Finite normal-form games.

(a) What are N, S and u in the definition of a *finite normal-form* (or, equivalently, *strategic-form*) *game* $G = \langle N, S, u \rangle$? **[1 pt]**

(b) Give the definition of a *strictly dominated* (pure or mixed) strategy in such a game. **[1 pt]**

(c) Give the definition of a *Nash equilibrium* (in pure or mixed strategies) in such a game. **[1 pt]**

(d) Find all pure strategies that are strictly dominated, and find all Nash equilibria (in pure or mixed strategies), in the game G with payoff bi-matrix

	a'	b'	c'
a	$3,4$	$0,0$	$0,5$
b	$0,0$	$0,1$	$1,0$
c	$4,0$	$1,0$	$0,1$

[3 pts]

Solution

- (a)-(c) *See Osborne-Rubinstein and lecture slides.*
- (d) Strictly dominated: a and a'. NE: $x_1 = x_2 = (0, 1/2, 1/2)$.

2. Two individuals, A and B, compete for a prize worth $V > 0$. If A makes effort $x > 0$ and B makes effort $y > 0$, then the the probability that A wins is $x/(x+y)$ and that B wins is $y/(x+y)$. [If no effort is made, then the probability of winning is zero.] How much effort will they each make if the disutility (or cost) of own effort is ax for A and by for B, where $a, b > 0$? Assume that each individual strives to maximize the expected value from winning the prize, net of the disutility of own effort. [That is, the probability of winning the prize times the value of the prize, minus the disutility of own effort].

(a) Write this up as a normal-form game $G = \langle N, S, u \rangle$. [Efforts are made simultaneously.] **[1 pt]**

(b) Draw a diagram indicating A's and B's best-reply curves. [That is, A's optimal effort, for each given effort by B, and B's optimal effort, for each given effort by A.] **[2 pts]**

(c) Prove that there exists a unique Nash equilibrium in pure strategies, and find this equilibrium. How do the equilibrium efforts depend on V? On a and b? Explain the intuition for your answers. **[2 pts]**

(d) Let $V = a = b = 1$. Find the Nash equilibrium effort pair. Does there exist pairs of effort levels $(x, y) > 0$ such that both A and B would be better off, than in the Nash equilibrium, if they could commit themselves to those effort levels? If such levels exist, specify such a pair, and explain the intuition why or why not such pairs of effort levels exist. **[2 pts]**

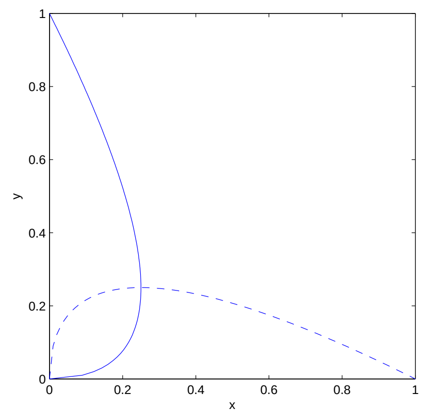
Solution (a) $N = \{1, 2\}$, $S_1 = S_2 = (0, +\infty)$ and

$$u_1(x, y) = \frac{Vx}{x+y} - ax \text{ and } u_2(x, y) = \frac{Vy}{x+y} - ay$$

(b) FOCs:

$$Vy = a(x+y)^2 \text{ and } Vx = b(x+y)^2$$

Best-reply curves:



(c) Unique NE: $x^* = bV/(a+b)^2$ and $y^* = aV/(a+b)^2$

(d) Yes. For $V = a = b = 1$ we have $x^* = y^* = 1/4$. Any $x = y \in (0, 1/4)$ is better than NE.

3. Consider the setup in Problem 2, now with $2a > b$, and assume that A first chooses effort $x > 0$. This is observed by B, who then chooses effort $y > 0$. Determine the effort levels in the game's subgame perfect equilibrium. **[4 pts]**

Solution

- Observing $x > 0$, B solves $\max_{y>0} \frac{Vy}{x+y} - by$. The second derivative of the goal function is negative, so the goal function is concave. The feasible set is open, so an optimum is found by solving the first-order condition:

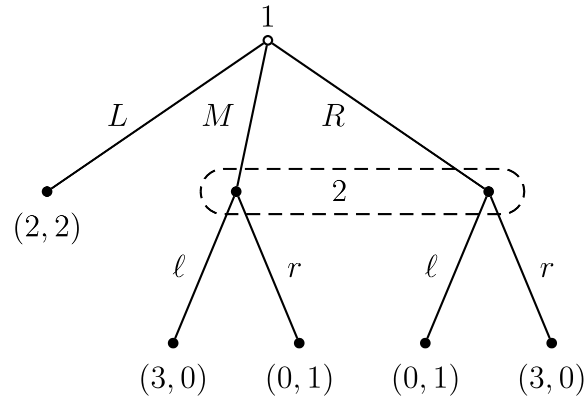
(1) $$Vx = b(x+y)^2.$$

  The best response is $y(x) = \sqrt{\frac{Vx}{b}} - x$, assuming this is positive (as it will be in equilibrium: that's why I assumed $2a > b$!).
- Anticipating this, A solves $\max_{x>0} \frac{Vx}{x+y(x)} - ax$, i.e., $\max_{x>0} \sqrt{Vbx} - ax$. Again, the goal function is concave. The feasible set is open, so an optimum is found by solving the first-order condition:

$$\frac{Vb}{2\sqrt{Vbx}} - a = 0 \quad \Leftrightarrow \quad x = \frac{Vb}{4a^2}.$$

- Substitution gives that $y(x) = \frac{V}{2a} - \frac{Vb}{4a^2} = \frac{V}{2a}\left(1 - \frac{b}{2a}\right)$. Both effort levels are positive under our assumption $b < 2a$.

4. Determine all sequential equilibria of the game below:



[8 pts]

Solution Ignoring player 1's trivial assessment in the singleton information set at the initial node, we can denote an assessment by

$$\begin{aligned}(\beta, \mu) &= (\beta_1, \beta_2, \mu) \\ &= (\underbrace{(\beta_1(\emptyset)(L), \beta_1(\emptyset)(M), \beta_1(\emptyset)(R))}_{\text{beh. str. of pl. 1 over } \{L,M,R\}}, \underbrace{(\beta_2(\{M,R\})(\ell), \beta_2(\{M,R\})(r))}_{\text{beh. str. of pl. 2 over } \{\ell,r\}}, \underbrace{(\mu(\{M,R\})(M), \mu(\{M,R\})(R))}_{\text{beliefs over } \{M,R\}}).\end{aligned}$$

For notational convenience, write $m := \mu(\{M,R\})(M)$, $q := \beta_2(\{M,R\})(\ell)$. The expected payoff to player 2 given assessment m is

$$m(1-q) + (1-m)q = m + q(1-2m).$$

We therefore consider three cases:

Case 1: $m < 1/2$: 2's unique best response is $q = 1$. Player 1's payoffs associated with L, M, R are

$$L \mapsto 2, \quad M \mapsto 3, \quad R \mapsto 0,$$

so M is the unique best response. But then player 2's beliefs are not consistent: there is no sequential equilibrium of this type.

Case 2: $m > 1/2$: 2's unique best response is $q = 0$. Player 1's payoffs associated with L, M, R are

$$L \mapsto 2, \quad M \mapsto 0, \quad R \mapsto 3,$$

so R is the unique best response. But then player 2's beliefs are not consistent: there is no sequential equilibrium of this type.

Case 3: $m = 1/2$: every $q \in [0,1]$ is a best response of player 2. Player 1's payoffs associated with L, M, R are

$$L \mapsto 2, \quad M \mapsto 3q, \quad R \mapsto 3(1-q).$$

- ⊠ Player 1 cannot choose both M and R with positive probability: sequential rationality would require that both are a best reply and therefore give the same payoff. This is possible only if $q = 1/2$, but then L gives a strictly higher payoff!
- ⊠ Player 1 cannot choose exactly one of the actions M and R with positive probability, since consistency would then lead to $m \in \{0, 1\}$.
- ⊠ So in sequential equilibrium, the only remaining possibility is that L is chosen with probability 1. To assure that this is a best response requires that $2 \geq 3q$ and $2 \geq 3(1-q)$, i.e., that $q \in [1/3, 2/3]$.
- ⊠ Our candidates for sequential equilibria are
$$\{(\beta_1, \beta_2, \mu) = ((1,0,0), (q, 1-q), (1/2, 1/2)) \mid q \in [1/3, 2/3]\}.$$
- ⊠ To verify consistency, notice that $((1,0,0), (q,1-q), (1/2,1/2))$ with $q \in [1/3, 2/3]$ is the limit of the sequence of assessments
$$((1-2/n, 1/n, 1/n), (q, 1-q), (1/2, 1/2)).$$

Part B – Combinatorial Game Theory

Jonas Sjöstrand

5. The *divisor game* is a two-player game with the following rules: From the beginning a number of positive integers are written on a blackboard. The players alternate moves and in each move the player at turn chooses one of the numbers on the blackboard and replaces it by any of its strictly smaller positive divisors. (A *divisor* of n is a positive integer d such that n/d is an integer.) For example, from the position $(2, 4, 6)$ the following positions can be reached in one move: $(1, 4, 6)$, $(2, 2, 6)$, $(2, 1, 6)$, $(2, 4, 3)$, $(2, 4, 2)$, and $(2, 4, 1)$. When all numbers on the blackboard are ones, no move is possible and as usual the player that cannot move is the loser.

(a) Let P_n denote the position with only the number n written on the blackboard. Find the Grundy value $g(P_n)$ for $1 \le n \le 8$. **[2 pts]**

(b) State a conjecture for the value of $g(P_n)$ for general n. **[1 pt]**

(c) Prove your conjecture. **[1 pt]**

(d) Find a winning move from the position $(126, 21\,870\,000, 16\,384\,000\,000)$. **[2 pts]**

Solution (a)

$$
\begin{aligned}
g(P_1) &= 0\\
g(P_2) &= \text{mex}\{g(P_1)\} = \text{mex}\{0\} = 1\\
g(P_3) &= \text{mex}\{g(P_1)\} = \text{mex}\{0\} = 1\\
g(P_4) &= \text{mex}\{g(P_1), g(P_2)\} = \text{mex}\{0, 1\} = 2\\
g(P_5) &= \text{mex}\{g(P_1)\} = \text{mex}\{0\} = 1\\
g(P_6) &= \text{mex}\{g(P_1), g(P_2), g(P_3)\} = \text{mex}\{0, 1, 1\} = 2\\
g(P_7) &= \text{mex}\{g(P_1)\} = \text{mex}\{0\} = 1\\
g(P_8) &= \text{mex}\{g(P_1), g(P_2), g(P_4)\} = \text{mex}\{0, 1, 2\} = 3
\end{aligned}
$$

(b) Conjecture: $g(P_n)$ is the number of factors in the prime factorization of n. Example: $600 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5$ has 6 prime factors.

(c) Let $\Omega(n)$ denote the number of prime factors of n. The divisors of n less than n are precisely the numbers we obtain by dropping some factors in the prime factorization of n. Thus, for any positive integer n we have

$$\{\Omega(P_d) \colon 1 \le d < n \text{ and } d \text{ divides } n\} = \{0, 1, \ldots, \Omega(n) - 1\}.$$

Now fix an n and suppose $g(P_k) = \Omega(k)$ for any $k < n$. Then it follows from the observation above that

$$
\begin{aligned}
g(P_n) &= \text{mex}\{g(P_d) \colon 1 \le d < n \text{ and } d \text{ divides } n\}\\
&= \text{mex}\{\Omega(P_d) \colon 1 \le d < n \text{ and } d \text{ divides } n\}\\
&= \text{mex}\{0, 1, \ldots, \Omega(n) - 1\} = \Omega(n),
\end{aligned}
$$

and the conjecture is true by induction.

(d) Prime factorization yields $126 = 2 \cdot 3^2 \cdot 7$, $21\,870\,000 = 2^4 \cdot 3^7 \cdot 5^4$, and $16\,384\,000\,000 = 2^{20} \cdot 5^6$ so $\Omega(126) = 4$, $\Omega(21\,870\,000) = 15$, and $\Omega(P_{16\,384\,000\,000}) = 26$, and we obtain

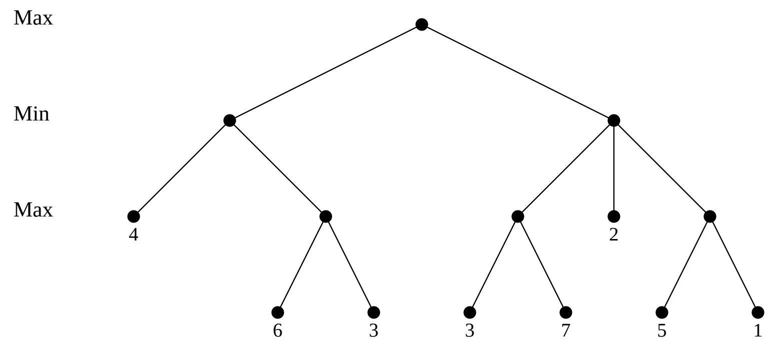
$$
\begin{aligned}
g(126, 21\,870\,000, 16\,384\,000\,000) &= g(P_{126} + P_{21\,870\,000} + P_{16\,384\,000\,000})\\
&= g(P_{126}) \oplus g(P_{21\,870\,000}) \oplus g(P_{16\,384\,000\,000})\\
&= \Omega(126) \oplus \Omega(21\,870\,000) \oplus \Omega(P_{16\,384\,000\,000})\\
&= 4 \oplus 15 \oplus 26\\
&= (100)_2 \oplus (1111)_2 \oplus (11010)_2\\
&= (10001)_2 = 17.
\end{aligned}
$$

Since $17 > 0$ there is a winning move, and it must be to replace the largest number by a divisor d such that $4 \oplus 15 \oplus \Omega(d) = 0$, that is $\Omega(d) = 4 \oplus 15 = 11$. For instance we could drop the 15 largest prime factors of 16 384 000 000 and choose $d = 2^{11} = 2048$.
So, one example of a winning move is to replace 16 384 000 000 by 2048.

6. Let $G = \{\, \{\, | \, 4, \, \{6, 3 \,|\, \}\, \}, \{\, | \, \{3, 7 \,|\, \}, 2, \{5, 1 \,|\, \}\} \,|\, \}$.

(a) Show that G is equal to a number and compute the value of G. **[2 pts]**

(b) If Left starts, G can be described by the following game tree:



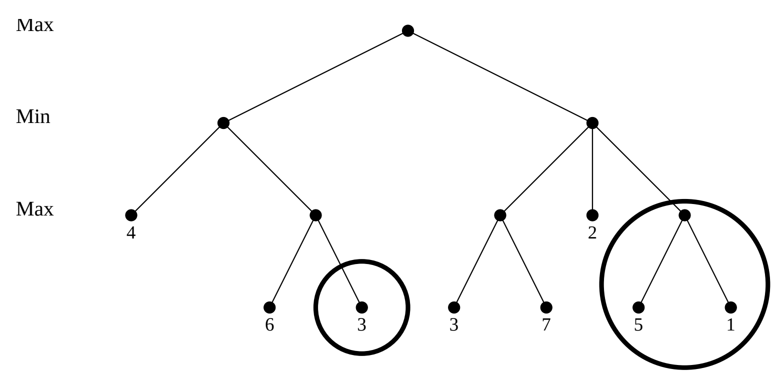
Perform a minimax search with alpha-beta pruning on the tree. The options of each subtree should be explored from left to right. Which parts of the tree are not explored by the search? **[3 pts]**

(c) What will be the outcome of the game if Left starts? Discuss why this is not the same number as the value of G. **[1 pt]**

Solution (a) We apply the simplicity rule repeatedly:

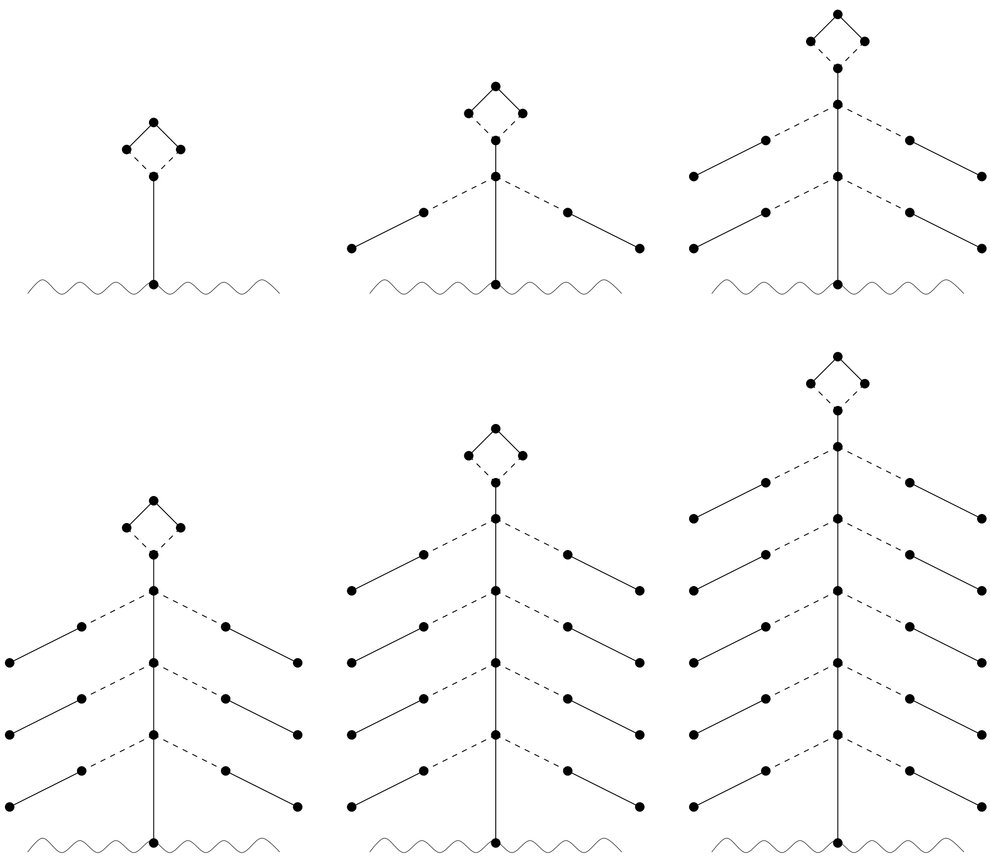
$G = \{\, \{\, | \, 4, \, \{6, 3 \,|\, \}\, \}, \{\, | \, \{3, 7 \,|\, \}, 2, \{5, 1 \,|\, \}\} \,|\, \} = \{\, \{\, | \, 4, 7 \,\}, \{\, | \, 8, 2, 6 \} \,|\, \} = \{\, 0, 0 \,|\, \} = 1.$

(b) The circled parts are not explored by the minimax search with alpha-beta pruning:



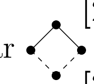
(c) The outcome will be 4 if Left starts. The reason this does not equal the game value 1 is that the game value takes into account what happens if the game is played together with other components in a sum of games, and in that situation the players do not need to alternate moves in G, nor does Left have to start.

7. Consider the infinite sequence $T_1, T_2, \dots$ of Christmas trees that begins with the following six trees and then continues in the obvious way.



Let G_n denote the Blue-Red Hackenbush game played on T_n with the solid edges coloured blue and the dashed ones coloured red (and the root connected to the ground as in the pictures).

(a) Compute the value of G_6. **[2 pts]**

(b) What happens to the game value if we remove the four-edge star  at the top of T_6? (Note that we keep the short edge below the star.) **[2 pts]**

(c) State a conjecture for the value of G_n for general n. **[1 pt]**

(d) Prove your conjecture. **[2 pts]**

Solution (a) A single four-edge star connected to the ground has value -1 as revealed for instance by a simple strategical argument. This means that $G_1 = 1 : -1 = \frac{1}{2}$. For $n \geq 2$ we have the recurrence relation $G_n = 1 : (G_{n-1} - 1)$ since each two-edge branch of a Christmas tree is worth $-1 : 1 = -\frac{1}{2}$ when connected to the ground. Thus,

$$
\begin{aligned}
G_1 &= \tfrac{1}{2},\\
G_2 &= 1 : (G_1 - 1) = 1 : -\tfrac{1}{2} = \tfrac{3}{4},\\
G_3 &= 1 : (G_2 - 1) = 1 : -\tfrac{1}{4} = \tfrac{7}{8},\\
G_4 &= 1 : (G_3 - 1) = 1 : -\tfrac{1}{8} = \tfrac{15}{16},\\
G_5 &= 1 : (G_4 - 1) = 1 : -\tfrac{1}{16} = \tfrac{31}{32},\\
G_6 &= 1 : (G_5 - 1) = 1 : -\tfrac{1}{32} = \tfrac{63}{64}.
\end{aligned}
$$

(b) Let G_n^* denote the game obtained by removing the star from T_n. Clearly, $G_1^* = 1$ and now the recurrence $G_n^* = 1 : (G_{n-1}^* - 1)$ yields $G_2^* = 1 : 0 = 1$, $G_3^* = 1 : 0 = 1$, and so on, so in particular $G_6^* = 1$.

(c) Conjecture: $G_n = 1 - 2^{-n}$ for all $n \geq 1$.

(d) The conjecture is true for $n = 1$, so suppose $n \geq 2$ and argue by induction over n. By the recurrence relation above, we have $G_n = 1 : (G_{n-1} - 1)$ which, by the induction hypothesis, equals $1 : -2^{-(n-1)}$. The sign-expansion of $-2^{-(n-1)} = -1 + \sum_{k=1}^{n-1} 2^{-k}$ is $-(+)^{n-1}$; in other words, to reach $-2^{-(n-1)}$ in Conway's number tree we should first go left and then right $n-1$ times. Thus, by definition of the colon operator, the sign-expansion of $1 : -2^{-(n-1)}$ is $+-(+)^{n-1}$. We conclude that

$$1 : -2^{-(n-1)} = 1 - \tfrac{1}{2} + \sum_{k=2}^{n} 2^{-k} = 1 - 2^{-n}$$

and hence $G_n = 1 - 2^{-n}$.

8. Let $G = \{\, \{8 \,|\, 7 \,||\, 1 \,|\, 0\} \mid \{-4 \,|\, -6\}, \{-2 \,|\, -4 \,||\, -9\} \,\}$.

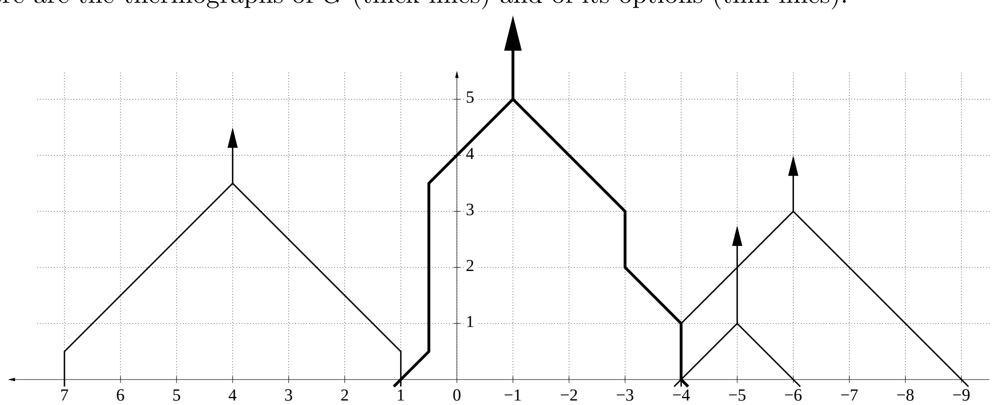
(a) Draw the thermograph of G. **[3 pts]**

(b) What is the temperature and mean value of G? **[1 pt]**

(c) Who will win G? **[1 pt]**

(d) Who will win the game $6G$? **[1 pt]**

Solution (a) Here are the thermographs of G (thick lines) and of its options (thin lines):



(b) The temperature is $t(G) = 5$ and the mean value is $G_\infty = -1$.

(c) Since the origin is strictly inside the thermograph we know that G is fuzzy to zero and hence the first player will win the game.

(d) By the mean-value theorem, $6G < 6G_\infty + t(G) + \varepsilon = -6 + 5 + \varepsilon < 0$ for sufficiently small positive ε, so Right will always win $6G$.