KTH Teknikvetenskap

SF2972 Game Theory
Exam with Solutions
March 15, 2013

Part A – Classical Game Theory

Jörgen Weibull and Mark Voorneveld

1. (a) What are N, S and u in the definition of a *finite normal-form* (or, equivalently, *strategic-form*) *game* $G = \langle N, S, u \rangle$? What is the mixed-strategy extension, $\tilde{G} = \langle N, \boxdot(S), \tilde{u} \rangle$ of such a game G? **[1 pt]**

(b) In terms of the mixed-strategy extension $\tilde{G}$ of an arbitrary finite game G: Give the definitions of a *strictly dominated* strategy, a *weakly dominated strategy*, a *Nash equilibrium*, and a *perfect equilibrium*. **[2 pts]**

(c) Consider the following finite two-player game G, representing price competition in a market where all consumers buy from the seller(s) with the lowest price. Both sellers have to simultaneously choose a price, p_1 and p_2, where $p_i \in P = \{0, 1, 2, 3, 4\}$. The profits to each seller are given in the payoff bi-matrix below, where seller 1 chooses row and seller 2 column. Find all *strictly dominated* pure strategies, all *weakly dominated* pure strategies, all pure-strategy *Nash equilibria*, and all pure-strategy *perfect equilibria*. **[2 pts]**

$p_1 \backslash p_2$	0	1	2	3	4
0	$-5, -5$	$-10, 0$	$-10, 0$	$-10, 0$	$-10, 0$
1	$0, -10$	$0, 0$	$0, 0$	$0, 0$	$0, 0$
2	$0, -10$	$0, 0$	$3, 3$	$6, 0$	$6, 0$
3	$0, -10$	$0, 0$	$0, 6$	$4, 4$	$8, 0$
4	$0, -10$	$0, 0$	$0, 6$	$0, 8$	$3, 3$

Solution (a) See book or lecture notes.

(b) See book or lecture notes.

(c) Strictly dominated strategies: $p_1 = 0$ and $p_2 = 0$. Weakly dominated strategies: $p_1 = 0, 1, 4$ and $p_2 = 0, 1, 4$. Pure-strategy Nash equilibria: $(p_1, p_2) = (1, 1)$ and $(p_1, p_2) = (2, 2)$. Pure-strategy perfect equilibrium: $(p_1, p_2) = (2, 2)$.

2. There are $n \geq 1$ partners who together own a firm. Each partner i chooses an effort level $x_i \geq 0$, resulting in total profit $g(y)$ for their firm, where y is the sum of all partners' efforts. The profit function $g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $g(0) = 0$ and it is twice differentiable with $g' > 0$, and $g'' \leq 0$. The profit is shared equally by the partners, and each partner's effort gives him or her (quadratic) disutility. The resulting utility level for each partner i is

$$u_i(x_1, .., x_n) = \frac{1}{n} g(x_1 + .. + x_n) - x_i^2/2$$

Each partner i has to decide his or her effort x_i without observing the others' efforts.

(a) Show that the game has exactly one Nash equilibrium (in pure strategies), and show that all partners make the same effort, x^*, in equilibrium. (A precise and formal argumentation is required.) Is the individual equilibrium effort x^* increasing or decreasing in n, or is it independent of n? Is the aggregate equilibrium effort, $y^* = nx^*$, increasing or decreasing in n, or is it independent of n? **[2 pts]**

(b) Suppose that the partners can pre-commit to a common effort level, the same for all. Let $\hat{x}$ be the common effort level that maximizes the sum of the partners' utilities. Characterize $\hat{x}$ in terms of an equation, and compare this level with the equilibrium effort x^* in (a), for $n = 1, 2, ...$ Are the partners better off now than in the equilibrium in (a)? How does this depend on n? Explain! **[1 pt]**

(c) Solve the tasks (a) and (b) explicitly for x^* and $\hat{x}$ in the special case when g is linear, $g(y) \equiv y$. **[2 pts]**

Solution (a) Each payoff function is twice differentiable and

$$\frac{\partial u_i}{\partial x_i} = \frac{1}{n} g'(x_1 + .. + x_n) - x_i \text{ and } \frac{\partial^2 u_i}{\partial x_i^2} = \frac{1}{n} g''(x_1 + .. + x_n) - 1 < 0$$

Hence u_i is strictly concave in x_i and $\partial u_i(x)/\partial x_i > 0$ when $x_i = 0$. Thus, a necessary and sufficient condition for x_i to be a best reply to the strategy profile $(x_1, .., x_n)$ is that $x_i > 0$ satisfies the first-order condition

$$\frac{1}{n} g'(x_1 + .. + x_n) = x_i$$

Since the left-hand side is the same for all i, we obtain that in Nash equilibrium $x_i = x^*$ where x^* satisfies

$$g'(nx^*) = nx^*.$$

The left-hand side is positive for $x^* = 0$, and it is continuous and non-increasing. Hence this equation has a unique solution. Moreover, the equation implies that nx^* is the same for all n, so aggregate equilibrium effort is independent of n, while individual equilibrium effort, x^*, is decreasing.

(b) The sum of all partners' utilities, when they all make the same effort z, is

$$W(z) = g(nz) - nz^2/2$$

Also this function is twice differentiable and strictly concave, and it has a positive derivative at $z = 0$, so a necessary and sufficient condition for a common effort level z to be optimal is that it satisfies $W'(z) = 0$, or, equivalently, $g'(nz) = z$. This equation has a unique solution, which we denote $\hat{x}$. We thus have

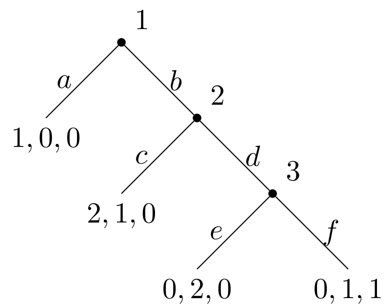
$$g'(n\hat{x}) = \hat{x}$$

Since $g'' \leq 0$, the individual effort, $\hat{x}$ is either independent of n (if $g'' = 0$) or decreasing in n (if $g'' < 0$). It also follows that $\hat{x} = x^*$ when $n = 1$ and $\hat{x} > x^*$ when $n > 1$. From this it also follows that individual utility is lower in Nash equilibrium than at the social optimum if and only if $n > 1$.

(c) In the special case when g is linear, $g(y) \equiv y$, we immediately obtain

$$x^* = 1/n \text{ and } \hat{x} = 1.$$

3. Find the pure-strategy subgame perfect equilibria of the game below:



[2 pts]

Solution

- In the final decision node, 3's payoff from e is 0 and from f is 1, so pl. 3 chooses f.
- Conditioning on this, 2's payoff from c is 1 and from d is 1, making 2 indifferent between the two actions.
- For pl. 1 in the initial node, we therefore distinguish two cases:
  (a) If 2 and 3 play (c, f), 1's payoff from a is 1 and from b is 2, so 1 chooses b: profile (b, c, f) is subgame perfect.
  (b) If 2 and 3 play (d, f), 1's payoff from a is 1 and from b is 0, so 1 chooses a: profile (a, d, f) is subgame perfect.

4. Use the deferred acceptance algorithm to find a stable matching in the marriage problem with ranking matrix:

	w_1	w_2	w_3	w_4
m_1	1, 3	2, 1	3, 4	4, 2
m_2	2, 2	1, 4	4, 2	3, 4
m_3	2, 4	4, 2	1, 3	3, 3
m_4	1, 1	2, 3	4, 1	3, 1

[1 pt]

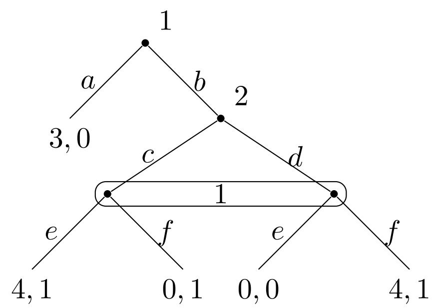
Solution The man-proposing variant ends after 4 iterations with match

$$(m_1, w_2), \quad (m_2, w_4), \quad (m_3, w_3), \quad (m_4, w_1),$$

and the woman-proposing variant ends after 3 iterations with match

$$(m_1, w_2), \quad (m_2, w_3), \quad (m_3, w_4), \quad (m_4, w_1).$$

5. Consider the following extensive form game:



(a) Find the corresponding strategic (i.e., normal form) game. **[1 pt]**
(b) Find all pure-strategy Nash equilibria. **[1 pt]**
(c) What is the outcome of iterated elimination of weakly dominated (pure) strategies? **[1 pt]**
(d) Find all subgame perfect equilibria in behavioral strategies. **[2 pts]**
(e) Find all sequential equilibria. **[2 pts]**

Solution (a)

	c	d
(a, e)	$3, 0$	$3, 0$
(a, f)	$3, 0$	$3, 0$
(b, e)	$4, 1$	$0, 0$
(b, f)	$0, 1$	$4, 1$

(b) $((b, e), c)$ and $((b, f), d)$.

(c) Consecutively eliminate:
- d: it is weakly dominated by c;
- $(a, e), (a, f), (b, f)$: they are now strictly dominated by (b, e).

Hence, only $((b, e), c)$ survives iterated elimination of weakly dominated strategies.

(d) Summarize a profile $b = (b_1, b_2)$ of behavioral strategies by:

$p_a = b_1(\emptyset)(a) \in [0, 1]$ the prob. 1 assigns to a in info set $\emptyset$

$p_e = b_1(\{(b, c), (b, d)\})(e) \in [0, 1]$ the prob. 1 assigns to e in info set $\{(b, c), (b, d)\}$

$p_c = b_2(\{b\})(c) \in [0, 1]$ the prob. 2 assigns to c in info set $\{b\}$

There are 2 subgames: the game itself and the proper subgame starting at 2's info set:

	c	d
e	$4, 1$	$0, 0$
f	$0, 1$	$4, 1$

Here, 1's set of best responses p_e to p_c is

$$\begin{cases} \{0\} & \text{if } p_c \in [0, 1/2), \\ [0, 1] & \text{if } p_c = 1/2, \\ \{1\} & \text{if } p_c \in (1/2, 1]. \end{cases}$$

and 2's set of best responses p_c to p_e is (note the weak dominance!)

$$\begin{cases} [0, 1] & \text{if } p_e = 0, \\ \{1\} & \text{if } p_e \in (0, 1] \end{cases}$$

So this subgame has equilibria

$$\begin{aligned} (p_e, p_c) &\in \{0\} \times [0, 1/2] \\ &\cup \{(1, 1)\} \end{aligned}$$

Now look at the game as a whole:

– If the players play $(p_e, p_c) \in \{0\} \times [0, 1/2]$ in the proper subgame, 1's payoff from a is 3 and from b is $4(1 - p_c)$, it follows that 1's set of best responses p_a is

$$\begin{cases} \{0\} & \text{if } p_c \in [0, 1/4), \\ [0,1] & \text{if } p_c = 1/4, \\ \{1\} & \text{if } p_c \in (1/4, 1/2]. \end{cases}$$

– If the players play $(p_e, p_c) = (1, 1)$ in the proper subgame, 1's payoff from a is 3 and from b is 4, so it is optimal to choose $p_a = 0$.

– Conclude that the subgame perfect equilibria in behavioral strategies are:

$$\begin{aligned} (p_a, p_c, p_e) \in\ & \{0\} \times [0, 1/4) \times \{0\} \\ & \cup [0, 1] \times \{1/4\} \times \{0\} \\ & \cup \{1\} \times (1/4, 1/2] \times \{0\} \\ & \cup \{(0, 1, 1)\}. \end{aligned}$$

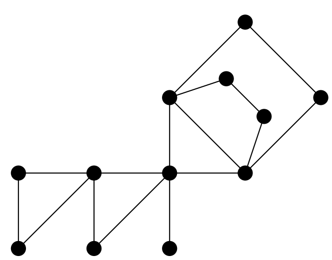
(e) The only nontrivial information set is $\{(b, c), (b, d)\}$ of player 2. So summarize a belief system by the probability $\alpha \in [0, 1]$ it assigns to the left node (b, c).
In a completely mixed profile of behavioral strategies, Bayes' Law gives $\alpha = p_c$. Taking limits, this equation has to hold in any consistent assessment.
If assessment (b, β) is a sequential equilibrium, then b is subgame perfect, so with the previous answer, the candidate sequential equilibria are behavioral strategies (p_a, p_c, p_e) as above and belief system $\alpha = p_c$.
Verifying that such assessments are both sequentially rational and consistent is standard.

Part B – Combinatorial Game Theory

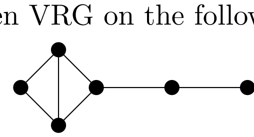
Jonas Sjöstrand

6. The *odd-odd vertex removal game* (odd-odd VRG) is an impartial two-player game played on an undirected graph. The players alternate moves, and in each move the player chooses a vertex of odd degree and removes it (and all its edges). When there are no odd-degree vertices left, no legal move is available and the player at turn will lose the game.

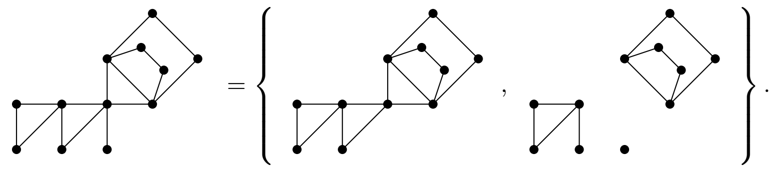
(a) Compute the Grundy value of the odd-odd VRG on the following graph. **[2 pts]**



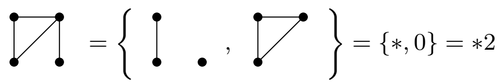
(b) A partizan variant of the game above is the *odd-even* VRG where Left removes vertices of odd degree and Right removes vertices of even degree. What is the canonical form of the odd-even VRG on the following graph? **[2 pts]**



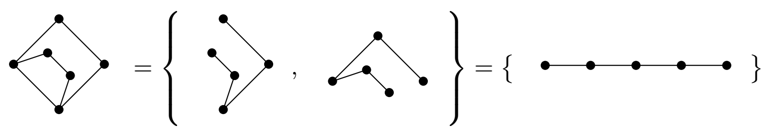
Solution (a) By a slight abuse of notation we will identify the odd-odd VRG played on a graph with the graph itself and write



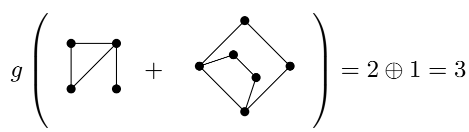
The first of these options is a terminal position because it has no vertices of odd degree, so its Grundy value is zero. The second option is a sum of three games, one of which is zero (the isolated vertex), the other ones being



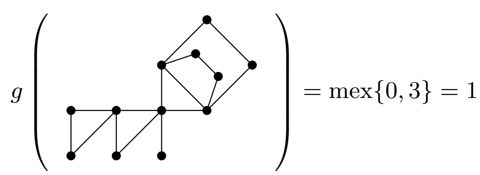
with Grundy value 2, and



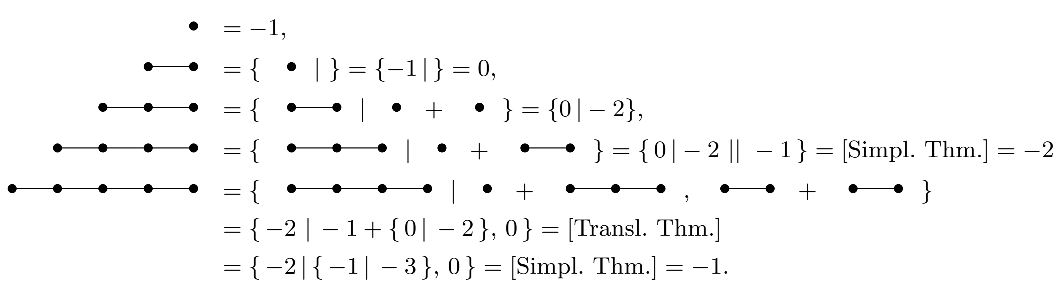
which has Grundy value $\text{mex}\{0\} = 1$ since the path of length four will terminate after an even number of moves (namely four) and hence is a $\mathcal{P}$-position.
We conclude that



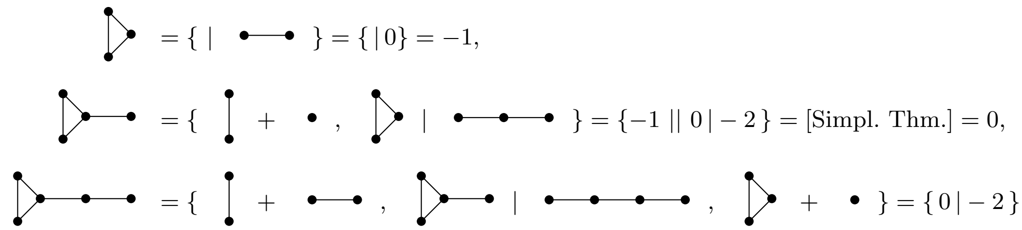
and thus



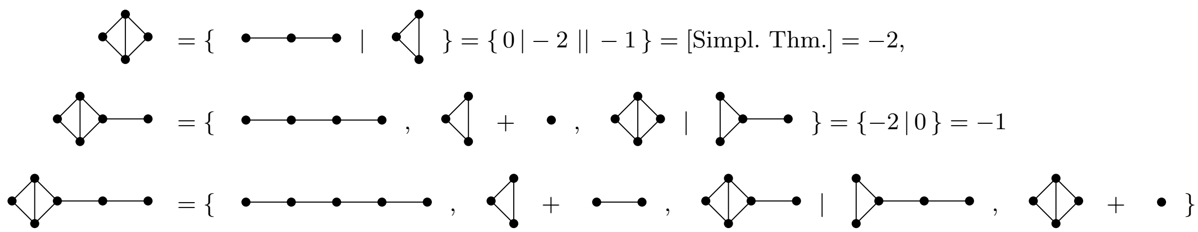
(b) We will probably have to consider all positions of the game, so let us start with paths up to length 4.



We proceed by computing positions with a triangle:



and finally we are ready to tackle the positions with a double triangle:



$= \{ -1 \mid \{ 0 \mid -2 \}, -3 \} = [\{ 0 \mid -2 \} \text{ is dominated by } -3] = \{ -1 \mid -3 \}.$

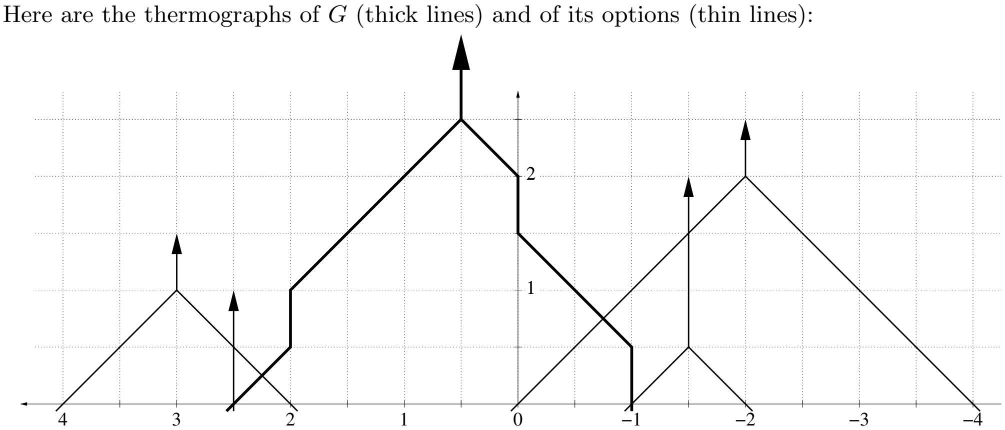
7. Let $G = \{ \frac{5}{2}, \{4 \mid 2\} \mid \{-1 \mid -2\}, \{0 \mid -4\} \}$.

(a) Draw the thermograph of G. **[2 pts]**

(b) What is the temperature and mean value of G? **[1 pt]**

(c) Who will win the game $-6G$? **[1 pt]**

Solution (a) Here are the thermographs of G (thick lines) and of its options (thin lines):



(b) The temperature is $t(G) = 5/2$ and the mean value is $G_\infty = 1/2$.

(c) By the Mean-Value Theorem, $6G > 6G_\infty - t(G) - \varepsilon = \frac{1}{2} - \varepsilon > 0$ for sufficiently small positive ε, so $-6G < 0$ and Right will always win $-6G$.

8. Answer the following questions and give proper motivations for your answers.

(a) Does there exist a game fuzzy to all integers? **[1 pt]**

(b) Does there exist a short game fuzzy to all integers? **[1 pt]**

(c) If x is a short number and G is a game not equal to a number, does it follow that $G + x = \{G^L + x \mid G^R + x\}$? **[1 pt]**

(d) If x is a number and G is a short game not equal to a number, does it follow that $G + x = \{G^L + x \mid G^R + x\}$? **[1 pt]**

(e) If $G^L < G^R$ for each left option G^L and each right option G^R of a game G, does it follow that G is equal to a number? **[1 pt]**

Solution (a) Yes, $G = \{\dots, -2, -1, 0, 1, 2, \dots \mid \dots, -2, -1, 0, 1, 2, \dots\}$ is fuzzy to any integer n, because $n \lhd G \lhd n$ since n is both a left option and a right option of G.

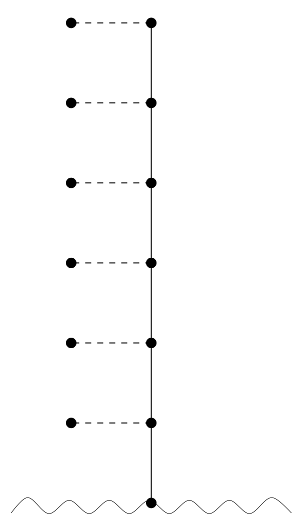
(b) No, since any short game is bounded by its number of positions. This can be seen as follows. If G has n positions, the game $G - n$ is a win for Right since he can always play in the $-n$ component. Hence $G < n$.

(c) No. If $G = \{\dots, -2, -1, 0, 1, 2, \dots \mid \dots, -2, -1, 0, 1, 2, \dots\}$ then $\{G^L + 1 \mid G^R + 1\} = G \neq G + 1$, and G is not equal to a number since it is fuzzy to zero. (Actually, as we saw above, it is fuzzy to all integers.)

(d) Yes, this is exactly the statement of the Translation Theorem (or the Strong Number Avoidance Theorem).

(e) No. Let $G = \{\, 0 \mid \uparrow \}$ where $\uparrow = \{\, 0 \mid * \}$. If G is equal to a number then $0 < G < \uparrow$, but that is impossible since $\uparrow$ is smaller than any positive number. Hence G is not equal to a number, but nevertheless $0 < \uparrow$.

9. Compute the value of the following Blue-Red Hackenbush position. (Solid edges are blue and dashed edges are red.) **[2 pts]**



Solution In Conway's number tree, if we go to the left once and then n times to the right we find the number -2^{-n}; this number has the sign expansion $-(+)^n$. If instead we go once to the right, once to the left, and then n times to the right, we find the number $1 - 2^{-(n+1)}$ with sign expansion $+-(+)^n$. By the definition of the colon operator, we deduce that $1 : -2^{-n} = 1 - 2^{-(n+1)}$ for any nonnegative integer n.

The depicted "ladder graph" — let us call it G_6 since it has 6 steps — equals $1 : (-1+G_5)$, where G_5 is a slightly smaller ladder graph with only 5 steps. More generally, $G_{n+1} = 1 : (-1 + G_n)$ for any positive integer n.

We claim that $G_n = 1 - 2^{-n}$ for any positive integer n and we will prove it by induction over n.

For $n = 1$ our claim is true since $G_1 = 1 : -1 = 1/2$. The induction step follows from the discussion above: $G_{n+1} = 1 : (-1 + G_n) = 1 : \bigl(-1 + (1 - 2^{-n})\bigr) = 1 : -2^{-n} = 1 - 2^{-(n+1)}$.

Thus, the value of the Hackenbush position is $G_6 = 1 - 2^{-6} = \frac{63}{64}$.