KTH Teknikvetenskap

SF2972 Game Theory
Exam with Solutions
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Part A – Classical Game Theory

Jörgen Weibull and Mark Voorneveld

1. Consider the two-player game G with normal form

	L	M	R
A	$8, 11$	$-3, 0$	$0, 0$
B	$9, -1$	$4, 1$	$0, 0$
C	$0, -2$	$0, 0$	$1, 4$

(a) For an arbitrary finite game in normal form: Give exact definitions of *weak* and *strict dominance* (for mixed strategies), *rationalizability* (of pure strategies), and *Nash equilibrium* (in mixed strategies).
(b) Find all pure strategies that are *strictly dominated* by a pure or mixed strategy.
(c) Find all *rationalizable* pure strategies.
(d) Find all *Nash equilibria*, in pure and mixed strategies.
(e) Define *perfect equilibrium* and find all such equilibria, in pure and mixed strategies.

[6 pts]

Solution
(a) See book and lecture slides.
(b) Strategy A. (Strictly dominated by a mixture of B and C.)
(c) Strategies B,C,M and R. (Strategy L is eliminated in the second round.)
(d) Strategy profiles (B,M), (C,R) and $x = (x_1, x_2)$ for $x_1 = \frac{4}{5}[B] + \frac{1}{5}[C]$ and $x_2 = \frac{1}{5}[M] + \frac{4}{5}[R]$.
(e) See book and lecture slides for definition. All three NE are perfect, since undominated and $n = 2$.

2. A group with $n \geq 1$ members together own a production unit. Each member i chooses an input level $x_i \geq 0$. Total output depends on total input, $x_1 + \cdots + x_n$. Output is a public good enjoyed by all group members and each member's input is costly to the member. The resulting utility level for each member i is

$$u_i(x_1, \ldots, x_n) = 2\sqrt{x_1 + \cdots + x_n} - x_i$$

Each member has to choose his or her input without observing the others' inputs.

(a) Show that the game has infinitely many Nash equilibria in pure strategies. (A precise and rigorous demonstration is required.) Solve for the aggregate input level (the sum of inidividual inputs in equilibrium). Is it increasing or decreasing in n, or independent of n? Explain your findings! **[2 pts]**

(b) Solve task (a) for two *altruistic* group members, that is, a group with two members who care (positively) about each others' utility. More exactly, let $n = 2$ and assume that group member i's (total) utility (with a capital "U") is given by

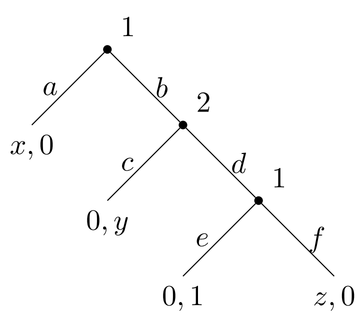
$$\begin{aligned} U_i(x_1, x_2) &= u_i(x_1, x_2) + \alpha_i \cdot u_j(x_1, x_2) \\ &= 2(1 + \alpha_i)\sqrt{x_1 + x_2} - x_i - \alpha_i x_j \end{aligned}$$

for $0 < \alpha_1 \leq \alpha_2 \leq 1$, the members' degrees of altruism, and for $i = 1, 2$ and $j \neq i$. **[2 pts]**

Solution (a) The sum of inputs is necessarily positive in equilibrium, since otherwise a unilateral deviation to a small positive input would increase the deviator's utility. Suppose first that *all* inputs are positive. A necessary condition for this to constitute an equilibrium is that the first-order condition $\partial u_i(x_1, \ldots, x_n) / \partial x_i = 0$ is met for all players i, which implies that $x_1 + \cdots + x_n = 1$. In fact, any such input profile $x = (x_1, \ldots, x_n)$ is a Nash equilibrium (even if some x_i are zero) since each player's payoff function is strictly concave in his/her input. The set of Nash equilibria is the unit simplex: $X^{NE} = \{x \in \mathbb{R}_+^n : x_1 + \cdots + x_n = 1\}$.

(b) It is still true that the sum of inputs has to be positive in equilibrium. Suppose that both inputs are positive. The necessary F.O.C. $\partial U_i(x_1, x_2) / \partial x_i = 0$, for $i = 1, 2$ gives $x_1 + x_2 = (1 + \alpha_i)^2$. If $\alpha_1 = \alpha_2 = \alpha$ then the same reasoning as in (a) leads to $X^{NE} = \{x \in \mathbb{R}_+^n : x_1 + x_2 = (1 + \alpha)^2\}$. If $\alpha_1 < \alpha_2$ then the F.O.C. cannot hold for both players. The unique Nash equilibrium is then that the less altruistic individual contributes nothing, $x_1 = 0$, and the more altruistic individual contributes $x_2 = (1 + \alpha_2)^2$. (This makes $\partial U_1(x_1, x_2) / \partial x_1 < 0$ and $\partial U_2(x_1, x_2) / \partial x_2 = 0$.)

3. Consider the game below:



Find all values of $x, y, z \in \mathbb{R}$ such that:

(a) $((a, e), c)$ is a Nash equilibrium. **[1 pt]**

(b) $((a, e), c)$ is a subgame perfect equilibrium. **[1 pt]**

Solution (a) The corresponding strategic form game is

	c	d
(a,e)	$x,0$	$x,0$
(a,f)	$x,0$	$x,0$
(b,e)	$0,y$	$0,1$
(b,f)	$0,y$	$z,0$

Hence, $((a,e),c)$ is a Nash equilibrium iff $x \geq 0$.

(b) Using backward induction, it follows that $((a,e),c)$ is a subgame perfect equilibrium iff $0 \geq z, y \geq 1, x \geq 0$.

4. Use the deferred acceptance algorithm to find a stable matching in the marriage problem with ranking matrix:

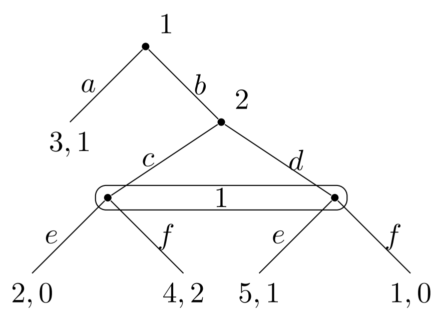
	w_1	w_2	w_3	w_4
m_1	$1,4$	$2,4$	$3,4$	$4,4$
m_2	$1,3$	$2,3$	$3,3$	$4,3$
m_3	$1,2$	$2,2$	$3,2$	$4,2$
m_4	$1,1$	$2,1$	$3,1$	$4,1$

[1 pt]

Solution The man- and woman-proposing variant both end after 4 iterations with match

$$(m_1, w_4), \quad (m_2, w_3), \quad (m_3, w_2), \quad (m_4, w_1).$$

5. Consider the following extensive form game:



(a) Find the corresponding strategic (i.e., normal form) game. **[1 pt]**
(b) Find all pure-strategy Nash equilibria. **[1 pt]**
(c) Find all subgame perfect equilibria in behavioral strategies. **[3 pts]**
(d) Find all sequential equilibria. **[2 pts]**

Solution 3(a)

	c	d
(a,e)	$3,1$	$3,1$
(a,f)	$3,1$	$3,1$
(b,e)	$2,0$	$5,1$
(b,f)	$4,2$	$1,0$

3(b) $((b,f),c)$ and $((b,e),d)$.

3(c) Summarize a profile $b = (b_1, b_2)$ of behavioral strategies by:

$p_a = b_1(\emptyset)(a) \in [0,1]$ the prob. 1 assigns to a in info set $\emptyset$

$p_e = b_1(\{(b,c),(b,d)\})(e) \in [0,1]$ the prob. 1 assigns to e in info set $\{(b,c),(b,d)\}$

$p_c = b_2(\{b\})(c) \in [0,1]$ the prob. 2 assigns to c in info set $\{b\}$

There are 2 subgames: the game itself and the proper subgame starting at 2's info set:

	c	d
e	2, 0	5, 1
f	4, 2	1, 0

Here, 1's set of best responses p_e to p_c is

$$\begin{cases} \{1\} & \text{if } p_c \in [0, 2/3), \\ [0,1] & \text{if } p_c = 2/3, \\ \{0\} & \text{if } p_c \in (2/3, 1]. \end{cases}$$

and 2's set of best responses p_c to p_e is

$$\begin{cases} \{1\} & \text{if } p_e \in [0, 2/3), \\ [0,1] & \text{if } p_e = 2/3, \\ \{0\} & \text{if } p_e \in (2/3, 1]. \end{cases}$$

So this subgame has equilibria

$$(p_e, p_c) \in \{(0,1), (1,0), (2/3, 2/3)\}.$$

Now look at the game as a whole:

- If the players play $(p_e, p_c) = (0,1)$ in the proper subgame, 1's payoff from a is 3 and from b is 4, so it is optimal to choose $p_a = 0$.
- If the players play $(p_e, p_c) = (1,0)$ in the proper subgame, 1's payoff from a is 3 and from b is 5, so it is optimal to choose $p_a = 0$.
- If the players play $(p_e, p_c) = (2/3, 2/3)$ in the proper subgame, 1's payoff from a is 3 and from b is 3, so any $p_a \in [0,1]$ is optimal.
- Conclude that the subgame perfect equilibria in behavioral strategies are:

$$(p_a, p_c, p_e) \in \{(0,1,0), (0,0,1)\} \cup ([0,1] \times \{2/3\} \times \{2/3\}).$$

3(d) The only nontrivial information set is $\{(b,c),(b,d)\}$ of player 2. So summarize a belief system by the probability $\alpha \in [0,1]$ it assigns to the left node (b,c).
In a completely mixed profile of behavioral strategies, Bayes' Law gives $\alpha = p_c$. Taking limits, this equation has to hold in any consistent assessment.
If assessment (b, β) is a sequential equilibrium, then b is subgame perfect, so with the previous answer, the candidate sequential equilibria are behavioral strategies (p_a, p_c, p_e) as above and belief system $\alpha = p_c$.
Verifying that such assessments are both sequentially rational and consistent is standard.

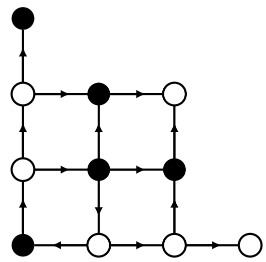
Part B – Combinatorial Game Theory

Jonas Sjöstrand

6. In a directed graph, a vertex with no outgoing edges is called a *sink*.

The *Sink Removal Game* is a partizan game played on a directed graph where each vertex is colored either black or white. The players alternate moves, and in each move the player chooses a sink and removes it (and all its ingoing edges). Left can only remove white sinks and Right can only remove black sinks. If no legal move is available, the player at turn will lose the game.

(a) Find the value of the following position in the Sink Removal Game. **[2 pts]**



(b) Show that the value of any position G in the Sink Removal Game is a number. Hint: Show that $G^L < G < G^R$ for any left option G^L and any right option G^R. **[3 pts]**

Solution (a) Note that the four vertices at the bottom left corner form a directed cycle and can never be removed in play, so the subgraph ●←○→●→○←●←○→○ induced by the remaining seven vertices is equivalent to the original game.
We have

(1) ●←○→●→○←●←○→○ =
{ ●←○→● + ●←○→○ , ●←○→●→○←●←○ | ○→●→○←●←○→○ }.

Let us expand the options one by one. First, we have

(2) ●←○→● = { | ●←○ }

and plugging

●←○ = { | ○} = { | 1 } = 0

into (2) yields

(3) ●←○→● = { | 0 } = −1.

Now, expand

(4) ●←○→○ = { ●←○ | ○→○ } = {0 | 2} = 1

and

(5) ●←○→●→○←●←○ = { ●←○→● + ●←○ | ○→●→○←●←○ } =
{ −1 + 0 | ○→●→○←●←○ }.

Plugging

○→●→○←●←○ = { ○→● + ●←○ | } = {0 + 0 | } = 1

into (5) yields

(6) ●←○→●→○←●←○ = { −1 | 1 } = 0.

Finally we expand

(7) ○→●→○←●←○→○ = { ○→● + ●←○→○ , ○→●→○←●←○ | } = { 0 + 1 , 1 | } = 2

and plug (3), (4), (6) and (7) into (1) to obtain

●←○→●→○←●←○→○ = { −1 + 1 , 0 | 2 } = 1.

The value of the game is 1.

(b) Let us show that if G is any position in the Sink Removal Game and G^L is a left option of G, then the game $G - G^L$ is greater than or equal to zero, that is, Left has a winning strategy if Right starts. The proof is by induction: Note that negating a game means interchanging black and white, and note also that once a vertex becomes a sink it remains so until it is removed. The game $-G^L$ is obtained from G by removing a white sink v and then interchanging black and white. Playing from $G - G^L$, if Right removes a black sink from the $-G^L$-component, Left counters by removing v from the G-component. If instead Right removes a black sink in the G-component, Left counters by removing the corresponding white sink in the $-G^L$ component.

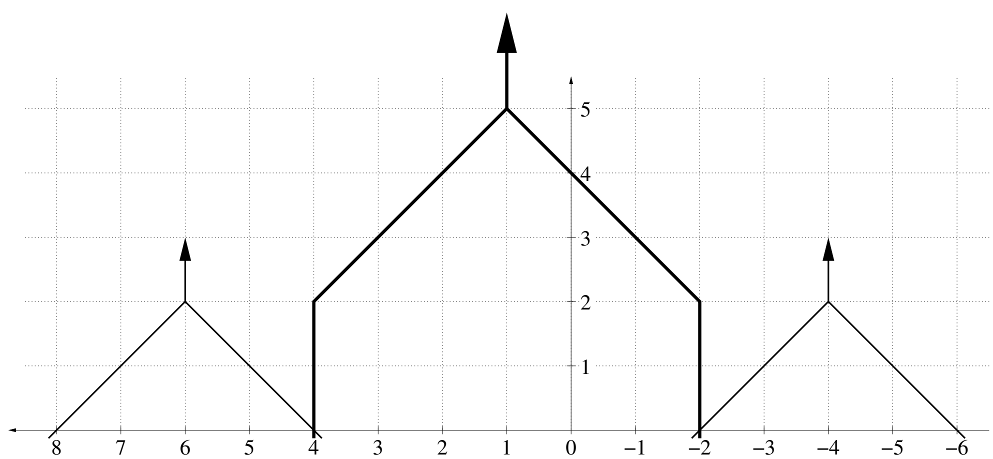
In either case, the resulting position is of the form $H - H^L$ for some game H and the induction hypothesis applies.
Analogously, it can be shown that $G^R - G \geq 0$, and since a game cannot be equal to one of its options, we have $G^L < G < G^R$ for any position G in the Sink Removal Game. By induction, we can assume that G^L and G^R are numbers and it follows that G is also a number.

7. Find a game G such that

- G has temperature 5,
- G has mean value 1,
- G is fuzzy to 3,
- G is fuzzy to -1, and
- $-3 < G < 5$.

Note that there might exist several G with these properties. You are only acquired to find one such game. **[4 pts]**

Solution One such game is $G = \{\, 8|4 \,||\, -2|-6 \,\}$ which has the following thermograph (where the thin lines are the thermographs of the options of G).



The numbers 3 and -1 are fuzzy to G since they lie strictly within the left and right boundaries. The numbers -3 and 5 on the other hand lie outside the fuzzy interval.

8. Consider the game of Nim with the additional rule that we are only allowed to remove one or four sticks in each move.

(a) Find the Grundy value $g(P_n)$ of a pile P_n of n sticks, for $0 \leq n \leq 9$. **[1 pt]**

(b) Find $g(P_n)$ for general n. **[1 pt]**

(c) Find a winning move from the three-pile position $(100, 49, 18)$. **[1 pt]**

Solution (a)

$$\begin{aligned}
g(P_0) &= \operatorname{mex} \emptyset = 0 \\
g(P_1) &= \operatorname{mex}\{g(P_{1-1})\} = \operatorname{mex}\{0\} = 1 \\
g(P_2) &= \operatorname{mex}\{g(P_{2-1})\} = \operatorname{mex}\{1\} = 0 \\
g(P_3) &= \operatorname{mex}\{g(P_{3-1})\} = \operatorname{mex}\{0\} = 1 \\
g(P_4) &= \operatorname{mex}\{g(P_{4-1}), g(P_{4-4})\} = \operatorname{mex}\{1, 0\} = 2 \\
g(P_5) &= \operatorname{mex}\{g(P_{5-1}), g(P_{5-4})\} = \operatorname{mex}\{2, 1\} = 0 \\
g(P_6) &= \operatorname{mex}\{g(P_{6-1}), g(P_{6-4})\} = \operatorname{mex}\{0\} = 1 \\
g(P_7) &= \operatorname{mex}\{g(P_{7-1}), g(P_{7-4})\} = \operatorname{mex}\{1\} = 0 \\
g(P_8) &= \operatorname{mex}\{g(P_{8-1}), g(P_{8-4})\} = \operatorname{mex}\{0, 2\} = 1 \\
g(P_9) &= \operatorname{mex}\{g(P_{9-1}), g(P_{9-4})\} = \operatorname{mex}\{1, 0\} = 2
\end{aligned}$$

(b) Let

$$f(n) := \begin{cases} 0 & \text{if } n \equiv 0 \text{ or } 2 \pmod 5, \\ 1 & \text{if } n \equiv 1 \text{ or } 3 \pmod 5, \\ 2 & \text{if } n \equiv 4 \pmod 5. \end{cases}$$

We will show that $g(P_n) = f(n)$ for all nonnegative n by induction over n. For $0 \leq n \leq 4$ we have already showed that $g(P_n) = f(n)$, so fix an $n \geq 5$ and assume that $g(P_k) = f(k)$ for any $0 \leq k \leq n-1$. Then,

$$g(P_n) = \operatorname{mex}\{g(P_{n-1}), g(P_{n-4})\} = \{\text{induction hypothesis}\} = \operatorname{mex}\{f(n-1), f(n-4)\} = f(n),$$

where the last equality is a simple matter of verification.

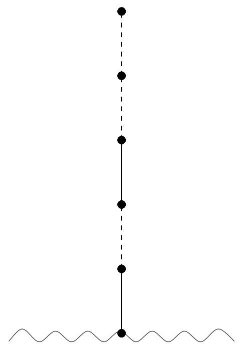
(c) For example, taking one stick from the middle pile is a winning move, since

$$g(P_{100} + P_{48} + P_{18}) = g(P_{100}) \oplus g(P_{48}) \oplus g(P_{18}) = 0 \oplus 1 \oplus 1 = 0.$$

9. (a) Construct a Blue-Red Hackenbush position that has value $18/32$. **[1 pt]**

(b) A *full binary tree* is an acyclic graph where each vertex has degree one or three. (The *degree* of a vertex is the number of edges incident to it. *Acyclic* means that the graph has no cycles.) Construct a Blue-Red Hackenbush position with value $5/8$ that is a full binary tree with one leaf (a vertex of degree one) attached to the ground. **[2 pts]**

Solution (a) The sign expansion of $18/32 = 9/16$ is $+-+--$ (that is, we can find $9/16$ in Conway's number tree by going right, left, right, left and left), so the following Blue-Red Hackenbush position has value $18/32$, where blue edges are solid and red edges are dashed.



(b) Since $5/8 = 1 : \big(-1 + (1 : -2)\big)$, the following full binary tree has value $5/8$.