KTH Teknikvetenskap

SF2972 Game Theory
Written Exam
March 17, 2011

Time: 14.00-19.00
No permitted aids
Examiner: Boualem Djehiche

The exam consists of two parts: Part A on classical game theory and Part B on combinatorial game theory. Each part will be scored from 0 to 25 points, so the maximal number of points you can get is 50. Each passed homework set handed in timely yields 1 bonus point. The bonus points are added to the points from the written exam and your grade is calculated as follows:

Points:	0-22	23-24	25-29	30-34	35-39	40-44	45-
Grade:	F	Fx	E	D	C	B	A

Write clearly and concisely, give precise definitions of game-theoretic concepts and precise statements of game-theoretic results that you refer to. Provide precise derivations and motivations for your answers.

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Part A – Classical Game Theory

Jörgen Weibull and Mark Voorneveld

1. Finite normal-form games.

(a) What are N, S and u in the definition of a *finite normal-form* (or, equivalently, *strategic-form*) *game* $G = \langle N, S, u \rangle$? **[1 pt]**

(b) Give the definition of a *strictly dominated* (pure or mixed) strategy in such a game. **[1 pt]**

(c) Give the definition of a *Nash equilibrium* (in pure or mixed strategies) in such a game. **[1 pt]**

(d) For finite and symmetric two-player games G: give the definition of an *evolutionarily stable* (pure or mixed) strategy. **[1 pt]**

2. Consider the two-player normal-form game G with payoff matrix

	a	b	c
a	$6,6$	$0,0$	$0,7$
b	$0,0$	$1,1$	$4,5$
c	$7,0$	$5,4$	$0,0$

.

(a) Find all pure strategies that are strictly dominated. **[1 pt]**

(b) Find all Nash equilibria in pure and/or mixed strategies. **[2 pts]**

(c) Find all evolutionarily stable strategies. **[1 pt]**

3. Two individuals, Al and Beth, contribute to a public good (say, a clean shared office) by making individual efforts $x \geq 0$ and $y \geq 0$. Individual utilities are given by

$$u_1(x,y) = (x+y)^a - ax^2 \quad \text{and} \quad u_2(x,y) = (x+y)^a - ay^2$$

for some $a \in (0,1)$. Each individual strives to maximize his or her utility.

(a) Game A: Suppose both effort levels are chosen simultaneously. Write up the normal form $G_A = \langle N, S, u \rangle$ of this game. Prove that there exists a unique Nash equilibrium in pure-strategies, and identify this equilibrium. **[2 pts]**

(b) Game B: Suppose that Al first chooses his effort level, x, and that this is observed by Beth, who then chooses her effort level, y. Write up the normal form $G_B = \langle N, S, u \rangle$ of this sequential game. **[1 pt]**

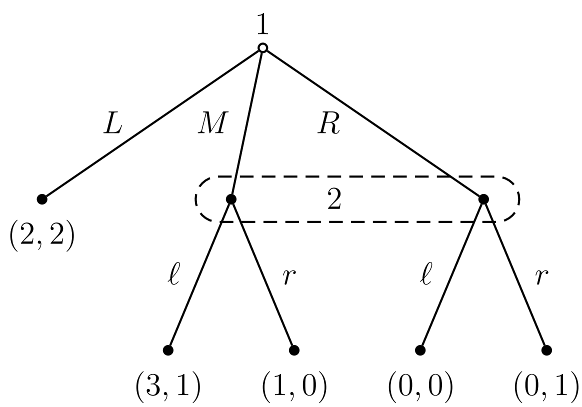
(c) Are the Nash equilibrium efforts in Game A, x^* and y^*, taken in any Nash equilibrium in Game B? **[2 pts]**

4. A child's action a (from a nonempty, finite set A) affects both her own private income $c(a)$ and her parents' income $p(a)$; for all $a \in A$ we have $0 \leq c(a) < p(a)$. The child is selfish: she cares only about the amount of money $c(a)$ she has. Her loving parents care both about how much money they have and how much their child has. Specifically, model the parents as a single player whose payoff equals the smaller of the amount of money the parents have and the amount of money the child has. The parents may transfer money to the child.

First the child takes an action $a \in A$. Then the parents observe the action and decide how much money $x \in [0, p(a)]$ to transfer to the child. The game ends with payoffs $c(a) + x$ to the child and $\min\{c(a) + x, p(a) - x\}$ to the parents.

Show that in a subgame perfect equilibrium the child takes an action that maximizes the sum of her private income and her parents' income. Not so selfish after all! **[4 pts]**

5. Consider the game below.



(a) Find the corresponding strategic form game. What is the outcome of iterated elimination of weakly dominated strategies (IEWDS)?

(b) Find all sequential equilibria. Compare the outcomes under (a) and (b): which do you find most reasonable?

[8 pts]

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Part B – Combinatorial Game Theory

Jonas Sjöstrand

6. Consider the game of Nim with the additional rule that we are only allowed to remove one stick or a prime number of sticks.

(a) Find the Grundy value $g(P_n)$ of a pile P_n of n sticks, for $0 \leq n \leq 8$. **[2 pts]**

(b) State a conjecture for the value of $g(P_n)$ for general n. **[1 pt]**

(c) Prove your conjecture. **[1 pt]**

(d) Find a winning move from the three-pile position $(100, 50, 25)$. **[1 pt]**

7. Alice and Bob plays the following game: First, Alice chooses a continent, then Bob chooses a country in that continent, and finally Alice chooses a city in that country. However, they may only choose continents, countries and cities from the following list of ten of the greatest cities in the world.

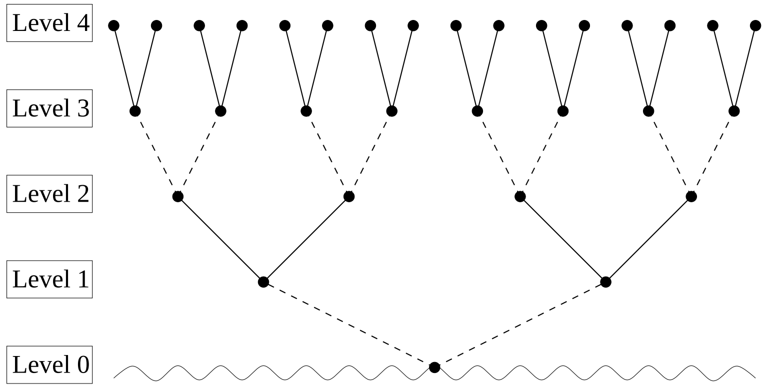
City	**Country**	**Continent**	**Population**/10^6
Tokyo	Japan	Asia	35.2
Jakarta	Indonesia	Asia	22.0
Bombay	India	Asia	21.3
New York	United States	America	20.6
São Paulo	Brazil	America	20.2
Mexico City	Mexico	America	18.7
Shanghai	China	Asia	18.4
Osaka	Japan	Asia	17.0
Calcutta	India	Asia	15.5
Los Angeles	United States	America	14.8

Alice wants to minimize the population of the chosen city, while Bob wants to maximize it. To compute the best strategy, Alice performs a complete minimax search on the game tree. When there are many possible choices, she decides to try them in alphabetical order.

(a) Draw the complete game tree. **[2 pts]**

(b) Circle the parts of the game tree that would not have been explored if Alice had used alpha-beta pruning. **[3 pts]**

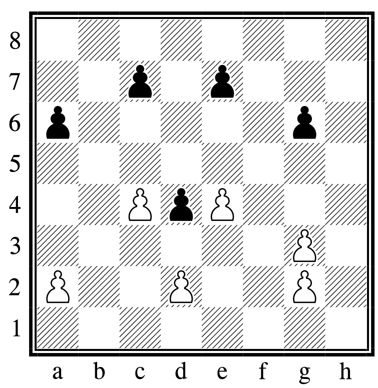
8. Let T_n be a binary tree of depth n with coloured edges such that edges between levels $i-1$ and i are blue if $n-i$ is even and red if $n-i$ is odd. For instance, T_4 looks like this, where the solid edges are blue and the dashed ones are red:



Let G_n denote the Blue-Red Hackenbush game played on T_n with the root connected to the ground.

(a) Compute the value of G_n for $0 \le n \le 7$. **[2 pts]**
(b) State a conjecture for the value of G_n for general n. **[1 pt]**
(c) Prove your conjecture. **[2 pts]**

9. Consider a simplified version of chess where there are only pawns and where the normal play convention is adopted. Show that the following position is equal to 1 if Left is white.



(Recall that a pawn can move forward one square if that square is unoccupied. "Forward" means up in the diagram for white pawns and down for black pawns. A white pawn at rank 2 (that is, row 2) has also the option of moving two squares up provided both squares above the pawn are unoccupied. Analogously, a black pawn at rank 7 has the option of moving two squares down. No captures are possible in our example.)

You may use the identity $\{0 \mid \uparrow\} = \uparrow + \uparrow + *$, where $\uparrow = \{0 \mid *\}$, without proving it.

[5 pts]

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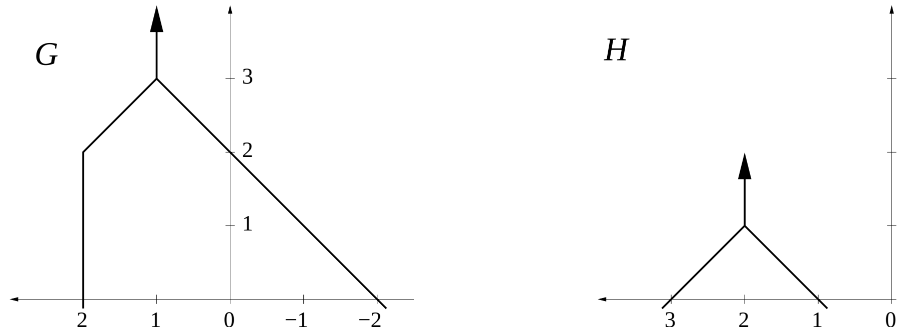
10. The games G and H have the following thermographs:



Which of the following graphs can possibly be the thermograph of $G+H$? (In each case, either give examples of G and H or prove that there are no such examples.) **[5 pts]**