KTH Teknikvetenskap

SF2972 Game Theory
Written Exam
June 10, 2011

Time: 14.00-19.00
No permitted aids
Examiner: Boualem Djehiche

The exam consists of two parts: Part A on classical game theory and Part B on combinatorial game theory. Each part will be scored from 0 to 25 points, so the maximal number of points you can get is 50. Each passed homework set handed in timely yields 1 bonus point. The bonus points are added to the points from the written exam and your grade is calculated as follows:

Points:	0-22	23-24	25-29	30-34	35-39	40-44	45-
Grade:	F	Fx	E	D	C	B	A

Write clearly and concisely, give precise definitions of game-theoretic concepts and precise statements of game-theoretic results that you refer to. Provide precise derivations and motivations for your answers.

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Part A – Classical Game Theory

Jörgen Weibull and Mark Voorneveld

1. Finite normal-form games.

(a) What are N, S and u in the definition of a *finite normal-form* (or, equivalently, *strategic-form*) *game* $G = \langle N, S, u \rangle$? **[1 pt]**

(b) Give the definition of a *strictly dominated* (pure or mixed) strategy in such a game. **[1 pt]**

(c) Give the definition of a *Nash equilibrium* (in pure or mixed strategies) in such a game. **[1 pt]**

(d) Find all pure strategies that are strictly dominated, and find all Nash equilibria (in pure or mixed strategies), in the game G with payoff bi-matrix

	a'	b'	c'
a	$3,4$	$0,0$	$0,5$
b	$0,0$	$0,1$	$1,0$
c	$4,0$	$1,0$	$0,1$

[3 pts]

2. Two individuals, A and B, compete for a prize worth $V > 0$. If A makes effort $x > 0$ and B makes effort $y > 0$, then the the probability that A wins is $x/(x+y)$ and that B wins is $y/(x+y)$. [If no effort is made, then the probability of winning is zero.] How much effort will they each make if the disutility (or cost) of own effort is ax for A and by for B, where $a, b > 0$? Assume that each individual strives to maximize the expected value from winning the prize, net of the disutility of own effort. [That is, the probability of winning the prize times the value of the prize, minus the disutility of own effort].

(a) Write this up as a normal-form game $G = \langle N, S, u \rangle$. [Efforts are made simultaneously.] **[1 pt]**

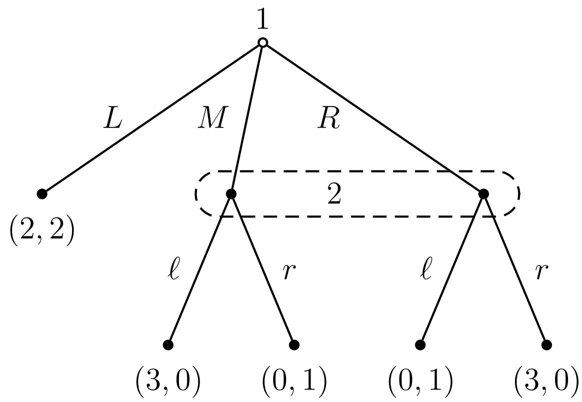
(b) Draw a diagram indicating A's and B's best-reply curves. [That is, A's optimal effort, for each given effort by B, and B's optimal effort, for each given effort by A.] **[2 pts]**

(c) Prove that there exists a unique Nash equilibrium in pure strategies, and find this equilibrium. How do the equilibrium efforts depend on V? On a and b? Explain the intuition for your answers. **[2 pts]**

(d) Let $V = a = b = 1$. Find the Nash equilibrium effort pair. Does there exist pairs of effort levels $(x, y) > 0$ such that both A and B would be better off, than in the Nash equilibrium, if they could commit themselves to those effort levels? If such levels exist, specify such a pair, and explain the intuition why or why not such pairs of effort levels exist. **[2 pts]**

3. Consider the setup in Problem 2, now with $2a > b$, and assume that A first chooses effort $x > 0$. This is observed by B, who then chooses effort $y > 0$. Determine the effort levels in the game's subgame perfect equilibrium. **[4 pts]**

4. Determine all sequential equilibria of the game below:



[8 pts]

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Part B – Combinatorial Game Theory

Jonas Sjöstrand

5. The *divisor game* is a two-player game with the following rules: From the beginning a number of positive integers are written on a blackboard. The players alternate moves and in each move the player at turn chooses one of the numbers on the blackboard and replaces it by any of its strictly smaller positive divisors. (A *divisor* of n is a positive integer d such that n/d is an integer.) For example, from the position $(2, 4, 6)$ the following positions can be reached in one move: $(1, 4, 6)$, $(2, 2, 6)$, $(2, 1, 6)$, $(2, 4, 3)$, $(2, 4, 2)$, and $(2, 4, 1)$. When all numbers on the blackboard are ones, no move is possible and as usual the player that cannot move is the loser.

(a) Let P_n denote the position with only the number n written on the blackboard. Find the Grundy value $g(P_n)$ for $1 \le n \le 8$. **[2 pts]**

(b) State a conjecture for the value of $g(P_n)$ for general n. **[1 pt]**

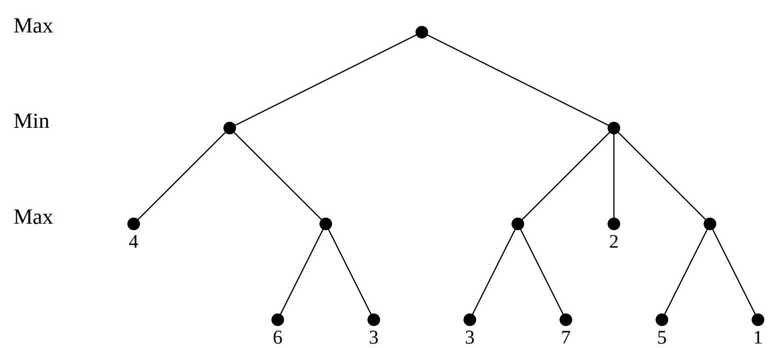
(c) Prove your conjecture. **[1 pt]**

(d) Find a winning move from the position $(126, 21\,870\,000, 16\,384\,000\,000)$. **[2 pts]**

6. Let $G = \{\,\{\,|\,4,\ \{6, 3\,|\,\}\,\}, \{\,|\,\{3, 7\,|\,\}, 2, \{5, 1\,|\,\}\}\,|\,\}$.

(a) Show that G is equal to a number and compute the value of G. **[2 pts]**

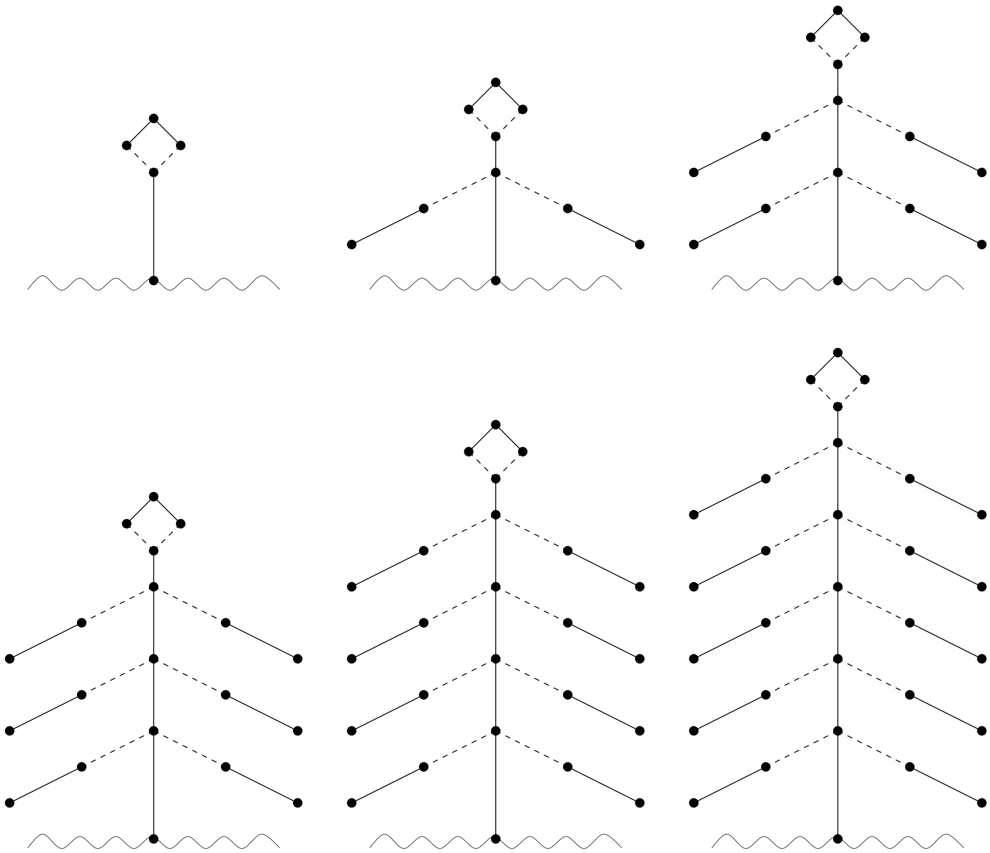
(b) If Left starts, G can be described by the following game tree:



Perform a minimax search with alpha-beta pruning on the tree. The options of each subtree should be explored from left to right. Which parts of the tree are not explored by the search? **[3 pts]**

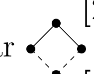
(c) What will be the outcome of the game if Left starts? Discuss why this is not the same number as the value of G. **[1 pt]**

7. Consider the infinite sequence $T_1, T_2, \ldots$ of Christmas trees that begins with the following six trees and then continues in the obvious way.



Let G_n denote the Blue-Red Hackenbush game played on T_n with the solid edges coloured blue and the dashed ones coloured red (and the root connected to the ground as in the pictures).

(a) Compute the value of G_6. **[2 pts]**

(b) What happens to the game value if we remove the four-edge star  at the top of T_6? (Note that we keep the short edge below the star.) **[2 pts]**

(c) State a conjecture for the value of G_n for general n. **[1 pt]**

(d) Prove your conjecture. **[2 pts]**

8. Let $G = \{\, \{8 \,|\, 7 \,||\, 1 \,|\, 0\} \mid \{-4 \,|\, -6\}, \{-2 \,|\, -4 \,||\, -9\} \,\}$.

(a) Draw the thermograph of G. **[3 pts]**

(b) What is the temperature and mean value of G? **[1 pt]**

(c) Who will win G? **[1 pt]**

(d) Who will win the game $6G$? **[1 pt]**