KTH Teknikvetenskap

SF2972 Game Theory
Exam
March 15, 2013

Time: 8.00-13.00
No permitted aids
Examiner: Boualem Djehiche

The exam is divided in two: Part A on classical game theory and Part B on combinatorial game theory. The maximal score is 20 points from part A and 15 points from part B, and your grade is calculated as follows:

Minimal score:	17	18	21	24	27	30
Grade:	Fx	E	D	C	B	A

Write clearly and concisely, give precise definitions of game-theoretic concepts and precise statements of game-theoretic results that you refer to. Provide precise derivations and motivations for your answers.

Part A – Classical Game Theory

Jörgen Weibull and Mark Voorneveld

1. (a) What are N, S and u in the definition of a *finite normal-form* (or, equivalently, *strategic-form*) *game* $G = \langle N, S, u \rangle$? What is the mixed-strategy extension, $\tilde{G} = \langle N, \boxdot(S), \tilde{u} \rangle$ of such a game G? **[1 pt]**

(b) In terms of the mixed-strategy extension $\tilde{G}$ of an arbitrary finite game G: Give the definitions of a *strictly dominated* strategy, a *weakly dominated strategy*, a *Nash equilibrium*, and a *perfect equilibrium*. **[2 pts]**

(c) Consider the following finite two-player game G, representing price competition in a market where all consumers buy from the seller(s) with the lowest price. Both sellers have to simultaneously choose a price, p_1 and p_2, where $p_i \in P = \{0, 1, 2, 3, 4\}$. The profits to each seller are given in the payoff bi-matrix below, where seller 1 chooses row and seller 2 column. Find all *strictly dominated* pure strategies, all *weakly dominated* pure strategies, all pure-strategy *Nash equilibria*, and all pure-strategy *perfect equilibria*. **[2 pts]**

$p_1 \backslash p_2$	0	1	2	3	4
0	$-5,-5$	$-10,0$	$-10,0$	$-10,0$	$-10,0$
1	$0,-10$	$0,0$	$0,0$	$0,0$	$0,0$
2	$0,-10$	$0,0$	$3,3$	$6,0$	$6,0$
3	$0,-10$	$0,0$	$0,6$	$4,4$	$8,0$
4	$0,-10$	$0,0$	$0,6$	$0,8$	$3,3$

2. There are $n \geq 1$ partners who together own a firm. Each partner i chooses an effort level $x_i \geq 0$, resulting in total profit $g(y)$ for their firm, where y is the sum of all partners' efforts. The profit function $g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $g(0) = 0$ and it is twice differentiable with $g' > 0$, and $g'' \leq 0$. The profit is shared equally by the partners, and each partner's effort gives him or her (quadratic) disutility. The resulting utility level for each partner i is

$$u_i(x_1, .., x_n) = \frac{1}{n} g(x_1 + .. + x_n) - x_i^2/2$$

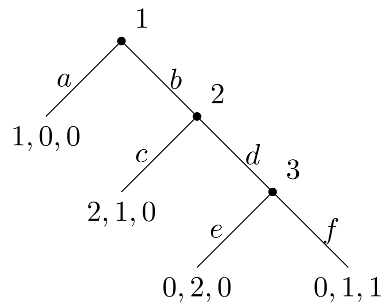
Each partner i has to decide his or her effort x_i without observing the others' efforts.

(a) Show that the game has exactly one Nash equilibrium (in pure strategies), and show that all partners make the same effort, x^*, in equilibrium. (A precise and formal argumentation is required.) Is the individual equilibrium effort x^* increasing or decreasing in n, or is it independent of n? Is the aggregate equilibrium effort, $y^* = nx^*$, increasing or decreasing in n, or is it independent of n? **[2 pts]**

(b) Suppose that the partners can pre-commit to a common effort level, the same for all. Let $\hat{x}$ be the common effort level that maximizes the sum of the partners' utilities. Characterize $\hat{x}$ in terms of an equation, and compare this level with the equilibrium effort x^* in (a), for $n = 1, 2, ...$ Are the partners better off now than in the equilibrium in (a)? How does this depend on n? Explain! **[1 pt]**

(c) Solve the tasks (a) and (b) explicitly for x^* and $\hat{x}$ in the special case when g is linear, $g(y) \equiv y$. **[2 pts]**

3. Find the pure-strategy subgame perfect equilibria of the game below:



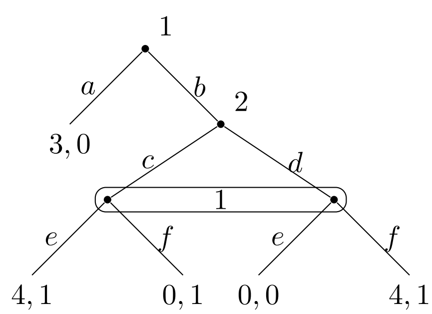
[2 pts]

4. Use the deferred acceptance algorithm to find a stable matching in the marriage problem with ranking matrix:

	w_1	w_2	w_3	w_4
m_1	1, 3	2, 1	3, 4	4, 2
m_2	2, 2	1, 4	4, 2	3, 4
m_3	2, 4	4, 2	1, 3	3, 3
m_4	1, 1	2, 3	4, 1	3, 1

[1 pt]

5. Consider the following extensive form game:



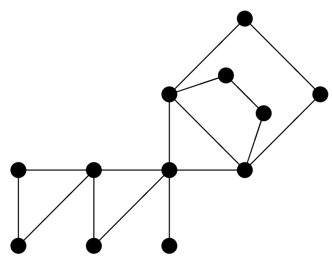
(a) Find the corresponding strategic (i.e., normal form) game. **[1 pt]**
(b) Find all pure-strategy Nash equilibria. **[1 pt]**
(c) What is the outcome of iterated elimination of weakly dominated (pure) strategies? **[1 pt]**
(d) Find all subgame perfect equilibria in behavioral strategies. **[2 pts]**
(e) Find all sequential equilibria. **[2 pts]**

Part B – Combinatorial Game Theory

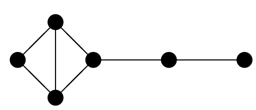
Jonas Sjöstrand

6. The *odd-odd vertex removal game* (odd-odd VRG) is an impartial two-player game played on an undirected graph. The players alternate moves, and in each move the player chooses a vertex of odd degree and removes it (and all its edges). When there are no odd-degree vertices left, no legal move is available and the player at turn will lose the game.

(a) Compute the Grundy value of the odd-odd VRG on the following graph. **[2 pts]**



(b) A partizan variant of the game above is the *odd-even* VRG where Left removes vertices of odd degree and Right removes vertices of even degree. What is the canonical form of the odd-even VRG on the following graph? **[2 pts]**



7. Let $G = \{\ \frac{5}{2}, \{4\,|\,2\}\ |\ \{-1\,|\,-2\}, \{0\,|\,-4\}\ \}$.

(a) Draw the thermograph of G. **[2 pts]**

(b) What is the temperature and mean value of G? **[1 pt]**

(c) Who will win the game $-6G$? **[1 pt]**

8. Answer the following questions and give proper motivations for your answers.

(a) Does there exist a game fuzzy to all integers? **[1 pt]**

(b) Does there exist a short game fuzzy to all integers? **[1 pt]**

(c) If x is a short number and G is a game not equal to a number, does it follow that $G + x = \{G^L + x \mid G^R + x\}$? **[1 pt]**

(d) If x is a number and G is a short game not equal to a number, does it follow that $G + x = \{G^L + x \mid G^R + x\}$? **[1 pt]**

(e) If $G^L < G^R$ for each left option G^L and each right option G^R of a game G, does it follow that G is equal to a number? **[1 pt]**

9. Compute the value of the following Blue-Red Hackenbush position. (Solid edges are blue and dashed edges are red.) **[2 pts]**