# SF3953 Markov Chains and Processes

Jimmy Olsson

Lecture 1
Basic definitions

16 February 2017

# Idag

General information

This lecture

# Today

General information

This lecture

# Schedule

- A preliminary schedule is as follows:

| Date | Hours | Room | Topic |
|---|---|---|---|
| 16/2 | 13-15 | 3418 | Basic definitions |
| 17/2 | 10-12 | 3418 | The strong Markov property |
| 1/3 | 15-17 | 3418 | Exercise class |
| 2/3 | 13-15 | 3418 | Atomic chains |
| 3/3 | 10-12 | 3418 | Minorisation and splitting |
| 15/3 | 15-17 | 3418 | Exercise class |
| 16/3 | 14-16 | 3721 | General irreducible chains |
| 17/3 | 13-15 | 3721 | Ergodic theory |
| 29/3 | 15-17 | 3418 | Exercise class |
| 30/3 | 15-17 | 3418 | Central limit theorems |
| 29/3 | 10-12 | 3418 | Geometric ergodicity |

- Course homepage:

`https://www.math.kth.se/matstat/gru/sf3953/`

# Literature

- The course will be based on lecture notes that will become available after each lecture.
- Recommended literature:

  Meyn, S. P. and Tweedie, R. L. (2009). *Markov Chains and Stochastic Stability*, 2nd Ed. Cambridge University Press, London.

  The first edition is available online; see the course homepage for a link.

# Examination

- The examination consists of two parts: home assignments and an oral.
- Solutions to the home assignments, which consist of sets of problems related to the theory presented in the lectures, are handed in at the beginning of each exercise class (except the last assignment).
- During the exercise classes the participants should be ready to present, on the blackboard, solutions to *all* the problems.
- Collaboration is allowed, but solutions are handed in *individually*.
- The oral treats all the theory developed in the lectures, and a course participant qualifies to the oral by performing sufficiently well on the home-assignment part of the course.

# Today

General information

This lecture

# Goal of this lecture

- Today we will
  - Recall some basic definitions and theorems,
  - Introduce transition kernels,
  - define homogeneous Markov chains and some related concepts.

# Thousands and thousands of Markov chains...

- In this course, focus is set on the *theory* of *general state-space Markov chains* (= discrete time). In particular, we are interested in *stochastic stability* (recurrence, stationarity, ergodicity).
- There is a rich literature on applications, including
  - *random walks*,
  - *population models* (birth-and-death processes, Wright-Fisher models and Galton-Watson processes, Ehrenfest's urn, ...),
  - *queueing and storage models*,
  - *time series analysis* (AR, INAR, self-exiting threshold AR, ARCH, ...),
  - *observation-driven models* (ARMA, GARCH, EGARCH, TGARCH, ...),
  - *Markov chain Monte Carlo methods* (Metropolis-Hastings sampler, Gibbs sampler, ...)

# Some background: filtrations

- We recall the following definitions.

### Definition (filtration)

A *filtration* of a measurable space $(\Omega, \mathcal{F})$ is an increasing sequence $\{\mathcal{F}_k : k \in \mathbb{N}\}$ of sub-$\sigma$-fields of $\mathcal{F}$.

### Definition (filtered probability space)

A *filtered probability space* $(\Omega, \{\mathcal{F}_k : k \in \mathbb{N}\}, \mathcal{F})$ is a probability space endowed with a filtration.

# Some background: stochastic processes

- Let $T$ be some set and $(\mathsf{X}, \mathcal{X})$ some measurable space.

### Definition (stochastic process)

A family of $\mathsf{X}$-valued random variables indexed by $T$ is called an $\mathsf{X}$-valued *stochastic process indexed by* $T$. (In this course, $T = \mathbb{N}$ or $T = \mathbb{Z}$.)

### Definition (adapted stochastic process)

A stochastic process $\{X_k : k \in \mathbb{N}\}$ is said to be *adapted* to the filtration $\{\mathcal{F}_k : k \in \mathbb{N}\}$ if for each $k \in \mathbb{N}$, $X_k$ is $\mathcal{F}_k$-measurable. (Notation: $\{(X_k, \mathcal{F}_k) : k \in \mathbb{N}\}$)

### Definition (natural filtration)

The *natural filtration* of a stochastic process $\{X_k : k \in \mathbb{N}\}$ defined on a probability space $(\Omega, \mathbb{P}, \mathcal{F})$ is the filtration $\{\mathcal{F}_k^X : k \in \mathbb{N}\}$ defined by

$$\mathcal{F}_k^X = \sigma(X_j : j \in \mathbb{N}, j \leq k), \quad k \in \mathbb{N}.$$

# Some background: monotone convergence

- During this first lecture, we will use repeatedly the following classical theorem.

**Theorem (monotone convergence)**

*Let* $(\mathsf{X}, \mathcal{X}, \mu)$ *be a measure space and* $\{f_n : n \in \mathbb{N}^*\}$ *be a sequence of measurable functions on* $\mathsf{X}$ *such that for all* $x \in \mathsf{X}$*,*

(i) $0 \leq f_1(x) \leq f_2(x) \leq \cdots \leq \infty$,

(ii) $f(x) = \lim_{n\to\infty} f_n(x)$.

*Then*

$$\int f(x)\,\mu(\mathrm{d}x) = \lim_{n\to\infty} \int f_n(x)\,\mu(\mathrm{d}x).$$

# Some background: monotone classes

**Definition**

Let $\Omega$ be a set. A collection $\mathcal{M}$ of subsets of $\Omega$ is called a *monotone class* if

(i) $\Omega \in \mathcal{M}$,

(ii) for $A \in \mathcal{M}$ and $B \in \mathcal{M}$ such that $A \subset B$, $B \setminus A \in \mathcal{M}$.

(iii) for all increasing sequences $\{A_n : n \in \mathbb{N}^*\}$ of sets in $\mathcal{M}$, $\cup_{n=1}^{\infty} A_n \in \mathcal{M}$.

**Theorem (the monotone class theorem)**

*Let $\mathcal{M}$ be a monotone class and assume that $\mathcal{E} \subset \mathcal{M}$ is stable by finite intersection. Then $\sigma(\mathcal{E}) \subset \mathcal{M}$.*

# Some background: the functional monotone class theorem

- In addition, the following theorem will be highly useful, in particular in combination with the monotone convergence theorem.

**Theorem (the functional monotone class theorem)**

*Let $\mathcal{H}$ be a vector space of bounded functions on $\Omega$ and $\mathcal{C}$ a class of subsets of $\Omega$ stable by finite intersection. Assume that $\mathcal{H}$ satisfies*

(i) $\mathbb{1}_\Omega \in \mathcal{H}$ *and for all* $A \in \mathcal{C}$, $\mathbb{1}_A \in \mathcal{H}$.

(ii) *If* $\{f_n, n \in \mathbb{N}\}$ *is an increasing sequence of functions of* $\mathcal{H}$ *such that* $\sup_{n\in\mathbb{N}} f_n = \lim_{n\to\infty} f_n = f$ *is bounded, then* $f \in \mathcal{H}$.

*Then* $\mathcal{H}$ *contains all the bounded* $\sigma(\mathcal{C})$*-measurable functions.*