SF3953 Markov Chains and Processes

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Lecture 2
Stopping times and the strong Markov property

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Today

This lecture

Goals of this lecture

- Today we will
  - discuss briefly the existence of Markov chains,
  - introduce some special stopping times of relevance for coming developments,
  - establish the strong Markov property.

Some background: the coordinate process

- For a given measurable space $(\mathsf{X}, \mathcal{X})$, let $\mathsf{X}^{\mathbb{N}}$ be the set of X-valued sequences $\omega = (\omega_0, \omega_1, \omega_2, \ldots)$.
- The σ-field $\mathcal{F} = \mathcal{X}^{\otimes \mathbb{N}}$ is generated by the algebra $\mathcal{A}$ of cylindrical sets of form

$$\prod_{n=0}^{\infty} A_n,$$

where $A_n \in \mathcal{X}$ for all $n \in \mathbb{N}$ and $A_n \neq \mathsf{X}$ for at most finitely many n.

Definition (coordinate process)

The *coordinate process* $\{X_k : k \in \mathbb{N}\}$ is the stochastic process defined on $(\mathsf{X}^{\mathbb{N}}, \mathcal{X}^{\otimes \mathbb{N}})$ by

$$X_k(\omega) = \omega_k, \quad \omega \in \mathsf{X}^{\mathbb{N}}.$$

- When $\{X_k : k \in \mathbb{N}\}$ is the coordinate process, we set $\mathcal{F}_n = \sigma(X_m : m \leq n)$.

Some background: stopping times

- Consider an adapted process $\{(X_k, \mathcal{F}_k) : k \in \mathbb{N}\}$. Define $\mathcal{F}_\infty$ as the σ-field generated by the union of all the $\{\mathcal{F}_k : k \in \mathbb{N}\}$.

Definition (stopping time)

(i) A random variable τ from Ω to $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ is called a *stopping time* if for all $k \in \mathbb{N}$, $\{\tau \leq k\} \in \mathcal{F}_k$.

(ii) The *stopping time σ-field* $\mathcal{F}_\tau$ is generated by the sets $A \subset \Omega$ such that for all $k \in \mathbb{N}$, $A \cap \{\tau \leq k\} \in \mathcal{F}_k$.

- It is easily checked that
  - $\mathcal{F}_\tau$ is indeed a σ-field,
  - a constant $\tau(\omega) = n \in \mathbb{N}$ is a stopping time (in which case $\mathcal{F}_\tau = \mathcal{F}_n$),
  - the event $\{\tau = \infty\}$ belongs to $\mathcal{F}_\infty$.

Some background: stopping times (cont'd)

- Given the stochastic process $\{X_k : k \in \mathbb{N}\}$ and some arbitrary $\mathcal{F}_\infty$-measurable random variable X_∞ we define

$$X_\tau = X_k \quad \text{on} \quad \{\tau = k\}, \quad k \in \bar{\mathbb{N}}.$$

- Note that X_τ is $\mathcal{F}_\tau$-measurable, since for $A \in \mathcal{F}_\tau$,

$$\begin{aligned}
\{X_\tau \in A\} \cap \{\tau \leq k\} \\
&= \bigcup_{\ell=0}^{k} \{X_\tau \in A\} \cap \{\tau = \ell\} \\
&= \bigcup_{\ell=0}^{k} \{X_\ell \in A\} \cap \{\tau = \ell\} \\
&= \bigcup_{\ell=0}^{k} \{X_\ell \in A\} \cap (\{\tau \leq \ell\} \setminus \{\tau \leq \ell - 1\}) \in \mathcal{F}_k.
\end{aligned}$$