SF3953: Markov Chains and Processes **Spring 2017**

Lecture 3: Atomic Chains and Irreducibility

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Goals of this lecture

- Introduce the concept of phi-irreducibility describing the communication between states and sets.
- Establish the transience-recurrence dichotomy for atomic chains.
- Introduce briefly Harris recurrence.

Irreducibility and transience/recurrence

For some given measurable space $(\mathsf{X}, \mathcal{X})$ and some Markov kernel P on $\mathsf{X} \times \mathcal{X}$, let $\{X_k : k \in \mathbb{N}\}$ be the canonical chain with kernel P (as defined last time).

Definition 3.1 (accessible set). *A set $A \in \mathcal{X}$ is said to be* accessible for the kernel P *(or, P-accessible) if $\mathbb{P}_x(\sigma_A < \infty) > 0$ for all $x \in \mathsf{X}$.*

Exercise 3.2. *Show that $A \in \mathcal{X}$ is accessible if $\mathbb{P}_x(\sigma_A < \infty) > 0$ for all $x \in A^{\complement}$.*

Definition 3.3 (phi-irreducibility). *The transition kernel P (or, alternatively, the Markov chain $\{X_k : k \in \mathbb{N}\}$ with transition kernel P) is said to be* phi-irreducible *if there exists $\phi \in \mathsf{M}_+(\mathcal{X})$ such all $A \in \mathcal{X}$ with $\phi(A) > 0$ are accessible. Such a measure is called an* irreducibility measure *for P.*

Example 3.4. *Assume that X is* countable. *In this case, we say that a state $x \in \mathsf{X}$* leads *to another state $y \in \mathsf{X}$, denoted $x \to y$, if $\mathbb{P}_x(\sigma_y < \infty) > 0$. If $x \to y$ and $y \to x$, x and y are said to* communicate, *which is denoted $x \leftrightarrow y$. The transition kernel (or chain) is called* irreducible *if $x \leftrightarrow y$ for all states $(x, y) \in \mathsf{X}^2$. Note that phi-irreducibility is weaker than this notion of irreducibility (since all measures on $\wp(\mathsf{X})$ are irreducibility measures if the chain is irreducible). The concepts are however not equivalent.*

Exercise 3.5. *Find an example of a chain on a countable state space that is phi-irreducible but not irreducible.*

In general there are many irreducibility measures. We will however show next that there exist *maximal irreducibility measures* ψ, which are such that any irreducibility measure ϕ is absolutely continuous with respect to ψ (i.e., for all $A \in \mathcal{X}$, $\psi(A) = 0 \Rightarrow \phi(A) = 0$). Our construction of a maximal irreducibility measure for P is based on the so-called *resolvent kernel associated with* P, which is, for $\eta \in (0,1)$, defined by

$$K_\eta : \mathsf{X} \times \mathcal{X} \ni (x, A) \mapsto (1-\eta) \sum_{n=0}^{\infty} \eta^n P^n(x, A).$$

Note that K_η is a Markov kernel for all $\eta \in (0,1)$.

Exercise 3.6.

(a) *Assume that* $\mu \in \mathsf{M}_+(\mathcal{X})$ *is invariant with respect to* P. *Show that* μ *is also invariant with respect to* K_η *for all* $\eta \in (0,1)$.

(b) *Show that for all* $A \in \mathcal{X}$ *and* $\eta \in (0,1)$,

$$\{x \in \mathsf{X} : \mathbb{P}_x(\sigma_A < \infty) > 0\} = \{x \in \mathsf{X} : K_\eta(x, A) > 0\}.$$

Theorem 3.7. *Let* P *be a transition kernel on* $\mathsf{X} \times \mathcal{X}$ *and let* ϕ *be an irreducibility measure for* P. *Then for all* $\eta \in (0,1)$, $\phi_\eta = \phi K_\eta$ *is a maximal irreducibility measure. In addition,*

$$A \in \mathcal{X} \text{ is accessible } \Leftrightarrow \phi_\eta(A) > 0. \tag{3.8}$$

Proof. To show that ϕ_η is an irreducibility measure, let $A \in \mathcal{X}$ be such that $\phi_\eta(A) > 0$. In addition, let

$$\bar{A} = \{x \in \mathsf{X} : \mathbb{P}_x(\sigma_A < \infty) > 0\} = \{x \in \mathsf{X} : K_\eta(x, A) > 0\},$$

where the equality holds by Exercise 3.6(b). If $\phi(\bar{A}) = 0$, then $K_\eta(\cdot, A) = 0$ ϕ-a.s., which implies that $\phi_\eta(A) = \phi K_\eta(A) = 0$. Thus, if $\phi_\eta(A) > 0$, then $\phi(\bar{A}) > 0$. Now, let $\bar{A}_m = \{x \in \mathsf{X} : \mathbb{P}_x(\sigma_A < \infty) > 1/m\}$, so that $\bar{A} = \cup_{m=1}^{\infty} \bar{A}_m$. Thus, there exists $m \in \mathbb{N}^*$ such that $\phi(\bar{A}_m) > 0$, and since ϕ is an irreducibility measure, $\bar{A}_m$ is accessible. Now, using the strong Markov property, for all $x \in \mathsf{X}$,

$$\begin{aligned} \mathbb{P}_x(\sigma_A < \infty) &\geq \mathbb{P}_x\left(\sigma_{\bar{A}_m} < \infty, \sigma_{\bar{A}_m} + \sigma_A \circ \sigma_{\bar{A}_m} < \infty\right) = \mathbb{P}_x\left(\sigma_{\bar{A}_m} < \infty, \sigma_A \circ \sigma_{\bar{A}_m} < \infty\right) \\ &= \mathbb{E}_x\left[\mathbb{1}_{\{\sigma_{\bar{A}_m} < \infty\}} \mathbb{E}_x\left[\mathbb{1}_{\{\sigma_A \circ \sigma_{\bar{A}_m} < \infty\}} \mid \mathcal{F}_{\sigma_{\bar{A}_m}}\right]\right] = \mathbb{E}_x\left[\mathbb{1}_{\{\sigma_{\bar{A}_m} < \infty\}} \mathbb{P}_{X_{\sigma_{\bar{A}_m}}}(\sigma_A < \infty)\right] \\ &\geq \frac{1}{m} \mathbb{P}_x\left(\sigma_{\bar{A}_m} < \infty\right) > 0, \end{aligned}$$

implying that ϕ_η is an irreducibility measure, which is also the sufficiency in the equivalence (3.8). To establish the other direction, note that for all $m \in \mathbb{N}$ and $A \in \mathcal{X}$, by the monotone convergence and Chapman-Kolmogorov theorems,

$$\begin{aligned}\int \phi_\eta(\mathrm{d}x)\, \eta^m P^m(x, A) &= (1-\eta) \int \phi(\mathrm{d}y) \int \sum_{\ell=0}^{\infty} \eta^\ell P^\ell(y, \mathrm{d}x)\, \eta^m P^m(x, A) \\ &= (1-\eta) \int \phi(\mathrm{d}y) \sum_{\ell=0}^{\infty} \eta^{\ell+m} P^{\ell+m}(y, A) \leq \phi K_\eta(A) = \phi_\eta(A),\end{aligned}$$

showing that $\phi_\eta K_\eta$ is absolutely continuous with respect to ϕ_η. Consequently, $\phi_\eta(A) = 0$ implies that $\phi_\eta K_\eta(A) = 0$ and, by definition, that $\phi_\eta(\bar{A}) = 0$. Thus,

$$\phi_\eta(\bar{A}) > 0 \Rightarrow \phi_\eta(A) > 0. \tag{3.9}$$

Hence if A is accessible, in which case $\bar{A} = \mathsf{X}$, then $\phi_\eta(A) > 0$, which is the necessity in (3.8).

Finally, let $\tilde{\phi} \in \mathsf{M}_+(\mathcal{X})$ be some other irreducibility measure and $A \in \mathcal{X}$ such that $\tilde{\phi}(A) > 0$. Then A is accessible and the by previous, $\phi_\eta(A) > 0$. Thus, $\tilde{\phi}$ is absolutely continuous with respect to ϕ_η, which completes the proof. □

For $A \in \mathcal{X}$, we define the *occupation time* η_A as the number of visits by $\{X_k : k \in \mathbb{N}\}$ to A, i.e.,

$$\eta_A = \sum_{k=0}^{\infty} \mathbb{1}_A(X_k) = \mathbb{1}_A(X_0) + \sum_{n=1}^{\infty} \mathbb{1}_{\{\sigma_A^{(n)} < \infty\}}.$$

Definition 3.10 (recurrence and uniform transience)**.** *A set $A \in \mathcal{X}$ is called* uniformly transient *if $\sup_{x \in A} \mathbb{E}_x[\eta_A] < \infty$. It is called* recurrent *if $\mathbb{E}_x[\eta_A] = \infty$ for all $x \in A$.*

For phi-irreducible transition kernels, the main result is the following *recurrence-transience dichotomy.*

Claim 3.11 (the recurrence-transience dichotomy)**.** *Let P be a phi-irreducible Markov kernel. Then either of the following statements holds true.*

(i) Every accessible set is recurrent, in which case we call P recurrent.

(ii) There is a countable cover of X with uniformly transient sets, in which case we call P transient.

During the coming lectures, we will establish Claim 3.11 under increasingly general assumptions. We will next establish Claim 3.11 in the particular case where the chain possesses an accessible *atom* (to be defined). Next time we will extend this result to the general case using *small*—"atom-like"—*sets* and the famous *splitting construction.*

Atomic chains

Definition 3.12. *A set $\boldsymbol{\alpha} \in \mathcal{X}$ is called an* atom *if there exists $\nu \in \mathsf{M}_1(\mathcal{X})$ such that $P(x, A) = \nu(A)$ for all $x \in \boldsymbol{\alpha}$ and $A \in \mathcal{X}$.*

Exercise 3.13. *Show that if $\boldsymbol{\alpha} \in \mathcal{X}$ is an atom for P, then it is also an atom for P^n for all $n \in \mathbb{N}$.*

For all $x \in \boldsymbol{\alpha}$, the common measure $P^n(x, \cdot)$ is denoted $P^n(\boldsymbol{\alpha}, \cdot)$. Similarly, we will write $\mathbb{P}_{\boldsymbol{\alpha}}$, etc.

The recurrence-transience dichotomy for atomic chains

We now establish the recurrence-transience dichotomy in the case of an accessible atom.

Theorem 3.14. *Assume that $\{X_k : k \in \mathbb{N}\}$ possesses an accessible atom $\boldsymbol{\alpha} \in \mathcal{X}$ with associated $\nu \in \mathsf{M}_1(\mathcal{X})$. Then the following holds true.*

(i) The chain is ϕ-irreducible, ν is an irreducibility measure, and a set $A \in \mathcal{X}$ is accessible if and only if $\mathbb{P}_{\boldsymbol{\alpha}}(\sigma_A < \infty) > 0$.

(ii) The atom $\boldsymbol{\alpha}$ is recurrent if and only if $\mathbb{P}_{\boldsymbol{\alpha}}(\sigma_{\boldsymbol{\alpha}} < \infty) = 1$ and (uniformly) transient otherwise; moreover, the chain is recurrent if $\boldsymbol{\alpha}$ is recurrent and transient otherwise.

Proof. To prove (i), write for $x \in \mathsf{X}$ and $A \in \mathcal{X}$, using the strong Markov property (last time),

$$\begin{aligned} \mathbb{P}_x(\sigma_A < \infty) &\geq \mathbb{P}_x(\sigma_{\boldsymbol{\alpha}} < \infty, \sigma_A \circ \theta_{\sigma_{\boldsymbol{\alpha}}} < \infty) \\ &= \mathbb{E}_x\left[\mathbb{1}_{\{\sigma_{\boldsymbol{\alpha}} < \infty\}} \mathbb{E}_x\left[\mathbb{1}_{\{\sigma_A \circ \theta_{\sigma_{\boldsymbol{\alpha}}} < \infty\}} \mid \mathcal{F}_{\sigma_{\boldsymbol{\alpha}}}\right]\right] \\ &= \mathbb{E}_x\left[\mathbb{1}_{\{\sigma_{\boldsymbol{\alpha}} < \infty\}} \mathbb{E}_{\boldsymbol{\alpha}}\left[\mathbb{1}_{\{\sigma_A < \infty\}}\right]\right] \\ &= \mathbb{P}_x\left(\sigma_{\boldsymbol{\alpha}} < \infty\right) \mathbb{P}_{\boldsymbol{\alpha}}\left(\sigma_A < \infty\right). \end{aligned} \tag{3.15}$$

Now, assume that $\mathbb{P}_{\boldsymbol{\alpha}}(\sigma_A < \infty) > 0$; then, since $\boldsymbol{\alpha}$ is accessible, the previous bound implies that $\mathbb{P}_x(\sigma_A < \infty) > 0$ for all $x \in \mathsf{X}$, which means that A is accessible. On the contrary, assume that A is accessible; then $\mathbb{P}_x(\sigma_A < \infty) > 0$ for all $x \in \mathsf{X}$, and in particular $\mathbb{P}_{\boldsymbol{\alpha}}(\sigma_A < \infty) > 0$. This proves the last claim of (i). To prove the first claim, note that

$$\mathbb{P}_x(\sigma_A < \infty) \geq \mathbb{P}_x\left(\sigma_{\boldsymbol{\alpha}} < \infty\right) \mathbb{P}_{\boldsymbol{\alpha}}\left(\sigma_A < \infty\right) \geq \mathbb{P}_x\left(\sigma_{\boldsymbol{\alpha}} < \infty\right) \mathbb{P}_{\boldsymbol{\alpha}}\left(X_1 \in A\right) = \mathbb{P}_x\left(\sigma_{\boldsymbol{\alpha}} < \infty\right) \nu(A).$$

Since $\boldsymbol{\alpha}$ is accessible, this implies that ν is a ϕ-irreducibility measure for P.

We turn to (ii). Recall the definition of the successive hitting times and note that

$$\mathbb{P}_{\boldsymbol{\alpha}}\left(\sigma_{\boldsymbol{\alpha}}^{(n)} < \infty\right) = \mathbb{P}_{\boldsymbol{\alpha}}\left(\sigma_{\boldsymbol{\alpha}}^{(n-1)} < \infty, \sigma_{\boldsymbol{\alpha}} \circ \theta_{\sigma_{\boldsymbol{\alpha}}^{(n-1)}} < \infty\right).$$

Thus, repeating the arguments of (3.15) yields for all $n \in \mathbb{N}^*$,

$$\mathbb{P}_{\boldsymbol{\alpha}}\left(\sigma_{\boldsymbol{\alpha}}^{(n)} < \infty\right) = \mathbb{P}_{\boldsymbol{\alpha}}\left(\sigma_{\boldsymbol{\alpha}}^{(n-1)} < \infty\right) \mathbb{P}_{\boldsymbol{\alpha}}\left(\sigma_{\boldsymbol{\alpha}} < \infty\right),$$

implying, by iteration, $\mathbb{P}_{\boldsymbol{\alpha}}(\sigma_{\boldsymbol{\alpha}}^{(n)} < \infty) = \mathbb{P}_{\boldsymbol{\alpha}}(\sigma_{\boldsymbol{\alpha}} < \infty)^n$. Consequently,

$$\mathbb{E}_{\boldsymbol{\alpha}}\left[\eta_{\boldsymbol{\alpha}}\right] = 1 + \sum_{n=1}^{\infty} \mathbb{P}_{\boldsymbol{\alpha}}\left(\sigma_{\boldsymbol{\alpha}} < \infty\right)^n = \sum_{n=0}^{\infty} \mathbb{P}_{\boldsymbol{\alpha}}\left(\sigma_{\boldsymbol{\alpha}} < \infty\right)^n,$$

from which we conclude that $\boldsymbol{\alpha}$ is recurrent if and only if $\mathbb{P}_{\boldsymbol{\alpha}}(\sigma_{\boldsymbol{\alpha}} < \infty) = 1$. Otherwise, $\boldsymbol{\alpha}$ is uniformly transient.

We turn to the second part of (ii). First note that for all $x \in \mathsf{X}$ and $A \in \mathcal{X}$,

$$\mathbb{P}_x\left(\sigma_A < \infty\right) = \sum_{\ell=1}^{\infty} \mathbb{P}_x\left(\sigma_A = \ell\right) \leq \sum_{\ell=1}^{\infty} \mathbb{P}_x\left(X_\ell \in A\right) = \sum_{\ell=1}^{\infty} P^\ell(x, A). \tag{3.16}$$

Assume that $\boldsymbol{\alpha}$ is recurrent. Recall that the chain is recurrent if all accessible sets are recurrent. Thus, let A be accessible and pick $x \in A$. Then by (3.16) there exists $s \in \mathbb{N}^*$ such that $P^s(x, \boldsymbol{\alpha}) > 0$. In addition, there exists $t \in \mathbb{N}^*$ such that $P^t(\boldsymbol{\alpha}, A) > 0$. Then, as $\boldsymbol{\alpha}$ is recurrent, using the Chapman-Kolmogorov theorem,

$$\begin{aligned}\mathbb{E}_x\left[\eta_A\right] \geq \sum_{n=1}^{\infty} P^{s+n+t}(x, A) &\geq \sum_{n=1}^{\infty} \int_{\boldsymbol{\alpha}} \int_{\boldsymbol{\alpha}} P^s(x, \mathrm{d}x')\, P^n(x', \mathrm{d}x'')\, P^t(x'', A) \\ &= P^s(x, \boldsymbol{\alpha}) P^t(\boldsymbol{\alpha}, A) \sum_{n=1}^{\infty} P^n(\boldsymbol{\alpha}, \boldsymbol{\alpha}) = \infty,\end{aligned}$$

showing that the chain is recurrent.

Now, assume that $\boldsymbol{\alpha}$ is transient. Then, since

$$\eta_{\boldsymbol{\alpha}} = \eta_{\boldsymbol{\alpha}} \circ \theta_{\tau_{\boldsymbol{\alpha}}} \mathbb{1}_{\{\tau_{\boldsymbol{\alpha}} < \infty\}},$$

using the strong Markov property, for all $x \in \mathsf{X}$,

$$\begin{aligned}\mathbb{E}_x\left[\eta_{\boldsymbol{\alpha}}\right] = \mathbb{E}_x\left[\mathbb{E}_x\left[\eta_{\boldsymbol{\alpha}} \circ \theta_{\tau_{\boldsymbol{\alpha}}} \mathbb{1}_{\{\tau_{\boldsymbol{\alpha}} < \infty\}} \mid \mathcal{F}_{\tau_{\boldsymbol{\alpha}}}\right]\right] &= \mathbb{E}_x\left[\mathbb{E}_{\boldsymbol{\alpha}}\left[\eta_{\boldsymbol{\alpha}}\right] \mathbb{1}_{\{\tau_{\boldsymbol{\alpha}} < \infty\}}\right] \\ &= \mathbb{P}_x\left(\tau_{\boldsymbol{\alpha}} < \infty\right) \mathbb{E}_{\boldsymbol{\alpha}}\left[\eta_{\boldsymbol{\alpha}}\right] \leq \mathbb{E}_{\boldsymbol{\alpha}}\left[\eta_{\boldsymbol{\alpha}}\right] < \infty. \end{aligned} \tag{3.17}$$

For all $j \in \mathbb{N}^*$, define $B_j = \{x \in \mathsf{X} : \sum_{\ell=1}^{j} P^\ell(x, \boldsymbol{\alpha}) \geq 1/j\}$; then, by (3.16), since $\boldsymbol{\alpha}$ is accessible, $\mathsf{X} = \cup_{j=1}^{\infty} B_j$. Now, note that for all $(j, n) \in {\mathbb{N}^*}^2$,

$$\begin{aligned}j \sum_{\ell=1}^{j} \int_{B_j} P^n(x, \mathrm{d}x')\, P^\ell(x', \boldsymbol{\alpha}) &= j \int_{B_j} P^n(x, \mathrm{d}x') \sum_{\ell=1}^{j} P^\ell(x', \boldsymbol{\alpha}) \\ &\geq P^n(x, B_j)\, j \inf_{x' \in B_j} \sum_{\ell=1}^{j} P^\ell(x', \boldsymbol{\alpha}) \geq P^n(x, B_j).\end{aligned}$$

On the other hand,

$$j \sum_{\ell=1}^{j} \int_{B_j} P^n(x, \mathrm{d}x')\, P^\ell(x', \boldsymbol{\alpha}) \leq j \sum_{\ell=1}^{j} \int P^n(x, \mathrm{d}x')\, P^\ell(x', \boldsymbol{\alpha}) = j \sum_{\ell=1}^{j} P^{n+\ell}(x, \boldsymbol{\alpha}).$$

Combining the last two inequalities yields, for all $x \in \mathsf{X}$,

$$\sum_{n=1}^{\infty} P^n(x, B_j) \leq j \sum_{\ell=1}^{j} \sum_{n=1}^{\infty} P^{n+\ell}(x, \boldsymbol{\alpha}) \leq j^2 \sum_{n=1}^{\infty} P^n(x, \boldsymbol{\alpha}),$$

where the right hand side is finite by (3.17). Hence, each B_j is transient, which completes the proof. □