SF3953: Markov Chains and Processes **Spring 2017**

Lecture 4: Atomic Chains and Invariant Measures

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Goals of this lecture

- Establish that chains with recurrent atoms admit invariant measures.
- Introduce a notion of small sets.

It turns out that in the phi-irreducible recurrent case, a Markov kernel admits a unique invariant measure.

Claim 4.1. *A phi-irreducible recurrent transition kernel (or Markov chain) admits a unique (up to a multiplicative constant) invariant measure.*

In the light of Claim 4.1, it is natural to ask whether the kernel admits an invariant *probability measure*. This leads to the following definition.

Definition 4.2 (positive/null chains)**.** *A phi-irreducible transition kernel (or Markov chain) is called* positive *if it admits an invariant probability measure; otherwise it is called* null.

During the coming lectures, we will establish Claim 4.1 under increasingly general assumptions and derive conditions for positivity. We will first consider the special case where the chain possesses an accessible atom.

Invariant measures and chains with accessible atoms

Let τ be a stopping time with respect to $\{\mathcal{F}_k : k \in \mathbb{N}\}$ and define the *occupation measure*

$$\lambda : \mathcal{X} \ni A \mapsto \mathbb{E}_\mu \left[\sum_{k=1}^{\tau} \mathbb{1}_A(X_k) \right].$$

Exercise 4.3.

(a) Show that for all τ and $\mu \in \mathsf{M}_1(\mathcal{X})$, $\lambda \in \mathsf{M}_+(\mathcal{X})$.

(b) Show that for all $f \in \mathsf{F}_+(\mathcal{X})$,

$$\lambda f = \mathbb{E}_\mu \left[\sum_{k=1}^{\tau} f(X_k) \right].$$

In the case where the chain possesses an accessible atom $\boldsymbol{\alpha}$, the occupation measure

$$\lambda_{\boldsymbol{\alpha}} : \mathcal{X} \ni A \mapsto \mathbb{E}_{\boldsymbol{\alpha}}\left[\sum_{k=1}^{\sigma_{\boldsymbol{\alpha}}} \mathbb{1}_A(X_k)\right]$$

is especially interesting.

Theorem 4.4. *Let $\boldsymbol{\alpha}$ be an accessible atom for the transition kernel P. Then $\lambda_{\boldsymbol{\alpha}}$ is sub-invariant with respect to P. It is invariant if and only if the atom $\boldsymbol{\alpha}$ is recurrent. In that case, any nontrivial P-invariant measure μ with $\mu(\boldsymbol{\alpha}) < \infty$ is proportional to $\lambda_{\boldsymbol{\alpha}}$, and $\lambda_{\boldsymbol{\alpha}}$ is a maximal irreducibility measure.*

Proof. In order to establish sub-invariance of $\lambda_{\boldsymbol{\alpha}}$, pick arbitrarily $A \in \mathcal{X}$ and write, using Exercise 4.3(b), the Markov property, and monotone convergence,

$$\begin{aligned}
\lambda_{\boldsymbol{\alpha}} P(A) &= \mathbb{E}_{\boldsymbol{\alpha}}\left[\sum_{n=1}^{\sigma_{\boldsymbol{\alpha}}} P(X_n, A)\right] = \mathbb{E}_{\boldsymbol{\alpha}}\left[\sum_{n=1}^{\sigma_{\boldsymbol{\alpha}}} \mathbb{E}_{X_n}\left[\mathbb{1}_A(X_1)\right]\right] \\
&= \mathbb{E}_{\boldsymbol{\alpha}}\left[\sum_{n=1}^{\sigma_{\boldsymbol{\alpha}}} \mathbb{E}_{\boldsymbol{\alpha}}\left[\mathbb{1}_A(X_1) \circ \theta_n \mid \mathcal{F}_n\right]\right] = \mathbb{E}_{\boldsymbol{\alpha}}\left[\sum_{n=1}^{\infty} \mathbb{1}_{\{\sigma_{\boldsymbol{\alpha}} \geq n\}} \mathbb{E}_{\boldsymbol{\alpha}}\left[\mathbb{1}_A(X_1) \circ \theta_n \mid \mathcal{F}_n\right]\right] \\
&= \sum_{n=1}^{\infty} \mathbb{E}_{\boldsymbol{\alpha}}\left[\mathbb{1}_{\{\sigma_{\boldsymbol{\alpha}} \geq n\}} \mathbb{1}_A(X_1) \circ \theta_n\right] = \mathbb{E}_{\boldsymbol{\alpha}}\left[\sum_{n=2}^{\sigma_{\boldsymbol{\alpha}}+1} \mathbb{1}_A(X_n)\right].
\end{aligned}$$

Now, since always

$$\begin{aligned}
\sum_{n=2}^{\sigma_{\boldsymbol{\alpha}}+1} \mathbb{1}_A(X_n) &= \sum_{n=1}^{\sigma_{\boldsymbol{\alpha}}} \mathbb{1}_A(X_n) - \mathbb{1}_A(X_1) + \mathbb{1}_{\{\sigma_{\boldsymbol{\alpha}} < \infty\}} \mathbb{1}_A(X_{\sigma_{\boldsymbol{\alpha}}+1}) \\
&= \sum_{n=1}^{\sigma_{\boldsymbol{\alpha}}} \mathbb{1}_A(X_n) - \mathbb{1}_A(X_1) + \mathbb{1}_{\{\sigma_{\boldsymbol{\alpha}} < \infty\}} \mathbb{1}_A(X_1) \circ \theta_{\sigma_{\boldsymbol{\alpha}}},
\end{aligned}$$

it holds that, using the strong Markov property,

$$\begin{aligned}
\lambda_{\boldsymbol{\alpha}} P(A) &= \lambda_{\boldsymbol{\alpha}}(A) - \mathbb{P}_{\boldsymbol{\alpha}}(X_1 \in A) + \mathbb{E}\left[\mathbb{1}_{\{\sigma_{\boldsymbol{\alpha}} < \infty\}} \mathbb{1}_A(X_1) \circ \theta_{\sigma_{\boldsymbol{\alpha}}}\right] \\
&= \lambda_{\boldsymbol{\alpha}}(A) - \mathbb{P}_{\boldsymbol{\alpha}}(X_1 \in A) + \mathbb{E}_{\boldsymbol{\alpha}}\left[\mathbb{1}_{\{\sigma_{\boldsymbol{\alpha}} < \infty\}} \mathbb{E}_{X_{\sigma_{\boldsymbol{\alpha}}}}\left[\mathbb{1}_A(X_1)\right]\right] \\
&= \lambda_{\boldsymbol{\alpha}}(A) - \mathbb{P}_{\boldsymbol{\alpha}}(X_1 \in A) + \mathbb{P}_{\boldsymbol{\alpha}}\left(\sigma_{\boldsymbol{\alpha}} < \infty\right) \mathbb{P}_{\boldsymbol{\alpha}}\left(X_1 \in A\right) \\
&= \lambda_{\boldsymbol{\alpha}}(A) - \mathbb{P}_{\boldsymbol{\alpha}}(X_1 \in A)\left[1 - \mathbb{P}_{\boldsymbol{\alpha}}\left(\sigma_{\boldsymbol{\alpha}} < \infty\right)\right],
\end{aligned}$$

which shows that $\lambda_{\boldsymbol{\alpha}}$ is sub-invariant for P. In addition, by Theorem 3.13(ii), $\lambda_{\boldsymbol{\alpha}}$ is invariant if and only if $\boldsymbol{\alpha}$ is recurrent.

To prove the second part, assume without loss of generality that $\mu(\boldsymbol{\alpha}) > 0$; otherwise, prove the claim for $\mu + \lambda_{\boldsymbol{\alpha}}$ for which $\mu(\boldsymbol{\alpha}) + \lambda_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}) \geq 1$ (indeed, if there exists $c > 1$ such that for all $A \in \mathcal{X}$, $\mu(A) + \lambda_{\boldsymbol{\alpha}}(A) = c\lambda_{\boldsymbol{\alpha}}(A)$, then $\mu(A) = \lambda_{\boldsymbol{\alpha}}(A)(c-1)$). Then, assume that $\mu(\boldsymbol{\alpha}) = 1$; otherwise, prove the claim for $\mu/\mu(\boldsymbol{\alpha})$.

We will next prove that $\lambda_{\boldsymbol{\alpha}}$ is minimal in the sense that for all invariant measures μ with $\mu(\boldsymbol{\alpha}) = 1$, $\mu \geq \lambda_{\boldsymbol{\alpha}}$. This follows if for all $n \in \mathbb{N}^*$ and $A \in \mathcal{X}$,

$$\mu(A) \geq \mathbb{E}_{\boldsymbol{\alpha}}\left[\sum_{k=1}^{\sigma_{\boldsymbol{\alpha}} \wedge n} \mathbb{1}_A(X_k)\right], \tag{4.5}$$

which we prove by induction. For $n = 1$, write

$$\mu(A) = \mu P(A) \geq \int_{\boldsymbol{\alpha}} P(x, A)\, \mu(\mathrm{d}x) = \mu(\boldsymbol{\alpha}) P(\boldsymbol{\alpha}, A) = P(\boldsymbol{\alpha}, A) = \mathbb{P}_{\boldsymbol{\alpha}}(X_1 \in A),$$

which equals the right hand side of (4.5) for $n = 1$.

Now, assume that (4.5) holds true for some $n \in \mathbb{N}^*$ and establish the same bound for $n \leftarrow n+1$. For this purpose, write, using the hypothesis and Exercise 4.3(b) (note that the right hand side of (4.5) is an occupation measure),

$$\begin{aligned}
\mu(A) &= \mu(\boldsymbol{\alpha}) P(\boldsymbol{\alpha}, A) + \int_{\boldsymbol{\alpha}^\complement} \mu(\mathrm{d}x)\, P(x, A) \\
&\geq P(\boldsymbol{\alpha}, A) + \mathbb{E}_{\boldsymbol{\alpha}}\left[\sum_{k=1}^{\sigma_{\boldsymbol{\alpha}} \wedge n} \mathbb{1}_{\boldsymbol{\alpha}^\complement}(X_k) P(X_k, A)\right] \\
&= P(\boldsymbol{\alpha}, A) + \sum_{k=1}^{n} \mathbb{E}_{\boldsymbol{\alpha}}\left[\mathbb{1}_{\{\sigma_{\boldsymbol{\alpha}} \geq k\}} \mathbb{1}_{\boldsymbol{\alpha}^\complement}(X_k) P(X_k, A)\right] \\
&= P(\boldsymbol{\alpha}, A) + \sum_{k=1}^{n} \mathbb{E}_{\boldsymbol{\alpha}}\left[\mathbb{1}_{\{\sigma_{\boldsymbol{\alpha}} \geq k+1\}} P(X_k, A)\right] \\
&= \ldots
\end{aligned}$$

Exercise 4.6. *Complete the induction step!*

Thus, $\mu \geq \lambda_{\boldsymbol{\alpha}}$.

We show that $\mu = \lambda_{\boldsymbol{\alpha}}$. Indeed, assume that there exists $A \in \mathcal{X}$ such that $\mu(A) > \lambda_{\boldsymbol{\alpha}}(A)$. However, for all $\eta \in (0, 1)$, by Exercise 3.6(a), μ and $\lambda_{\boldsymbol{\alpha}}$ are both invariant with respect to K_η and by Exercise 3.6(b), $K_\eta(x, \boldsymbol{\alpha}) > 0$ for all $x \in \mathsf{X}$. Thus,

$$\begin{aligned}
1 = \mu(\boldsymbol{\alpha}) = \mu K_\eta(\boldsymbol{\alpha}) &= \int_A K_\eta(x, \boldsymbol{\alpha})\, \mu(\mathrm{d}x) + \int_{A^\complement} K_\eta(x, \boldsymbol{\alpha})\, \mu(\mathrm{d}x) \\
&> \int_A K_\eta(x, \boldsymbol{\alpha})\, \lambda_{\boldsymbol{\alpha}}(\mathrm{d}x) + \int_{A^\complement} K_\eta(x, \boldsymbol{\alpha})\, \lambda_{\boldsymbol{\alpha}}(\mathrm{d}x) = \lambda_{\boldsymbol{\alpha}} K_\eta(\boldsymbol{\alpha}) = \lambda_{\boldsymbol{\alpha}}(\alpha) = 1,
\end{aligned}$$

which is a contradiction.

Finally, we show that $\lambda_{\boldsymbol{\alpha}}$ is a maximal irreducibility measure. Let ϕ be an irreducibility measure and put, for some $\eta \in (0,1)$, $\phi_\eta = \phi K_\eta$. By Theorem 3.7, ϕ_η is a maximal irreducibility measure. We assume $A \in \mathcal{X}$ is such that $\phi_\eta(A) = 0$ and show that $\lambda_{\boldsymbol{\alpha}}(A) = 0$, implying that $\lambda_{\boldsymbol{\alpha}}$ is an irreducibility measure. Nevertheless, in the proof of Theorem 3.7 it was shown that $\phi_\eta K_\eta$ was absolutely continuous with respect to ϕ_η. Thus, $\phi_\eta(A) = 0$ implies that $\phi_\eta K_\eta(A) = 0$, implying in turn, by Exercise 3.6(b), that $\mathbb{P}_x(\sigma_A < \infty) = 0$ for ϕ_η-a.a. $x \in \mathsf{X}$. In particular $\mathbb{P}_x(\sigma_A < \infty) = 0$ for ϕ_η-a.a. $x \in \boldsymbol{\alpha}$, which, since $\mathbb{P}_{\boldsymbol{\alpha}}(\sigma_A < \infty)$ is constant and $\phi_\eta(\boldsymbol{\alpha}) > 0$, implies that $\mathbb{P}_{\boldsymbol{\alpha}}(\sigma_A < \infty) = 0$. Now,

$$\lambda_{\boldsymbol{\alpha}}(A) = \mathbb{E}_{\boldsymbol{\alpha}}\left[\mathbb{1}_{\{\sigma_A = \infty\}} \sum_{k=1}^{\sigma_{\boldsymbol{\alpha}}} \mathbb{1}_A(X_k)\right] = \sum_{k=1}^{\infty} \mathbb{E}_{\boldsymbol{\alpha}}\left[\mathbb{1}_{\{\sigma_A = \infty\}} \mathbb{1}_{\{\sigma_{\boldsymbol{\alpha}} \le k\}} \mathbb{1}_A(X_k)\right] = 0,$$

from which it follows that $\lambda_{\boldsymbol{\alpha}}$ is an irreducibility measure. By Theorem 3.7, $\lambda_{\boldsymbol{\alpha}} K_\eta$ is a maximal irreducibility measure. On the other hand, by Exercise 3.6(a), $\lambda_{\boldsymbol{\alpha}}$ is invariant with respect to K_η, implying that $\lambda_{\boldsymbol{\alpha}} = \lambda_{\boldsymbol{\alpha}} K_\eta$ is maximal.

The proof is complete. □