SF3953: Markov Chains and Processes	**Spring 2017**
Lecture 6: General Phi-irreducible Chains	
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Goals of this lecture

- To extend the recurrence-transience dichotomy to general phi-irreducible chains via the split chain.
- Establish, using again the split chain, that general recurrent phi-irreducible chains admit a unique invariant measure.

The recurrence-transience dichotomy for general phi-irreducible chains

In the following, let $(\mathsf{X}, \mathcal{X})$ be a given measurable space. Next, we will use the splitting construction introduced in Lecture 5 for establishing the recurrence-transience dichotomy for general phi-irreducible chains.

Theorem 6.1. *Let P be a phi-irreducible transition kernel on $\mathsf{X} \times \mathcal{X}$ that admits an accessible $(1, \varepsilon, \nu)$-small set $C \in \mathcal{X}$. Then P is either recurrent or transient. It is recurrent if and only if the small set C is recurrent.*

Proof. Since the split chain possesses an accessible atom, it is, by Theorem 3.14, phi-irreducible and either recurrent or transient. By Exercise 5.12, for all $B \in \mathcal{X}$ and $x \in \mathsf{X}$,

$$\check{\mathbb{E}}_{\delta_x^*}\left[\eta_{B \times \{0,1\}}\right] = \mathbb{E}_x\left[\eta_B\right]. \tag{6.2}$$

Assume first that the split chain is recurrent. Note that for all $\check{A} \in \check{\mathcal{X}}$ and $x \in \mathsf{X}$, using Proposition 2.4 and the definition (5.6) of a split measure,

$$\check{\mathbb{E}}_{\delta_x^*}[\eta_{\check{A}}] = \int \check{\mathbb{E}}_{\check{y}}\left[\eta_{\check{A}}\right] \delta_x^*(\mathrm{d}\check{y}) = (1-\varepsilon)\mathbb{1}_C(x)\check{\mathbb{E}}_{(x,0)}\left[\eta_{\check{A}}\right] + \mathbb{1}_{C^\complement}(x)\check{\mathbb{E}}_{(x,0)}\left[\eta_{\check{A}}\right] + \varepsilon \mathbb{1}_C(x)\check{\mathbb{E}}_{(x,1)}\left[\eta_{\check{A}}\right]. \tag{6.3}$$

Let B be an accessible set for P. By Theorem 5.22, $B \times \{0,1\}$ is accessible for $\check{P}$ and by recurrence of the split chain it holds, for all $x \in B$, using (6.3) with $\check{A} = B \times \{0,1\}$, that $\check{\mathbb{E}}_{\delta_x^*}[\eta_{B\times\{0,1\}}] = \infty$. Hence, by (6.2), $\mathbb{E}_x[\eta_B] = \infty$. Thus, B is recurrent for P, which implies that P is recurrent.

Now, assume that the split chain is transient. Then by Theorem 3.14, the atom $\check{\boldsymbol{\alpha}}$ is transient. As in the proof of Theorem 3.14, define $B_j = \{x \in \mathsf{X} : \sum_{\ell=1}^{j} \check{P}^\ell((x,0), \check{\boldsymbol{\alpha}}) \geq 1/j\}$ for all $j \in \mathbb{N}^*$.

Exercise 6.4. *Show $B_j \times \{0,1\}$ is transient for all $j \in \mathbb{N}^*$.*

Now by (6.2) and (6.3), each B_j is transient for P and since the atom $\check{\boldsymbol{\alpha}}$ is accessible, $\mathsf{X} = \cup_{j=1}^\infty B_j$. We conclude that P is transient.

It remains to establish the last statement. We have shown that P is recurrent if and only of $\check{P}$ is so. Moreover, by Theorem 3.14, $\check{P}$ is recurrent if and only if $\check{\boldsymbol{\alpha}}$ is so. It is hence enough to show that $\check{\boldsymbol{\alpha}}$ is recurrent if and only if C is so. For $x \in C$, using the Markov property, Exercise 5.8, and (6.2),

$$
\begin{aligned}
\check{\mathbb{E}}_{\delta_x^*}\left[\eta_{\check{\boldsymbol{\alpha}}}\right] &= \sum_{k=0}^{\infty} \check{\mathbb{E}}_{\delta_x^*}\left[\mathbb{1}_C(X_k)\mathbb{1}_{\{D_k=1\}}\right] \\
&= \varepsilon + \sum_{k=1}^{\infty} \check{\mathbb{E}}_{\delta_x^*}\left[\mathbb{1}_C(X_k)\mathbb{1}_{\{D_k=1\}}\right] = \varepsilon + \sum_{k=1}^{\infty} \check{\mathbb{E}}_{\delta_x^*}\left[\check{\mathbb{E}}_{\check{X}_{k-1}}\left[\mathbb{1}_C(X_1)\mathbb{1}_{\{D_1=1\}}\right]\right] \\
&= \varepsilon + \sum_{k=1}^{\infty} \check{\mathbb{E}}_{\delta_x^*}\left[\check{\mathbb{E}}_{\check{X}_{k-1}}\left[\mathbb{1}_C(X_1)\check{\mathbb{P}}_{\check{X}_{k-1}}\left(D_1 = 1 \mid X_1\right)\right]\right] = \varepsilon + \varepsilon \sum_{k=1}^{\infty} \check{\mathbb{E}}_{\delta_x^*}\left[\mathbb{1}_C(X_k)\right] \\
&= \varepsilon \check{\mathbb{E}}_{\delta_x^*}\left[\eta_{C\times\{0,1\}}\right] = \varepsilon \mathbb{E}_x\left[\eta_C\right]. \quad (6.5)
\end{aligned}
$$

Thus, if C is recurrent there exists, by (6.5) and (6.3) (use the latter identity with $\check{A} = \check{\boldsymbol{\alpha}}$), an $\check{x} \in C \times \{0,1\}$ such that $\mathbb{E}_{\check{x}}[\eta_{\check{\boldsymbol{\alpha}}}] = \infty$. But since

$$\mathbb{E}_{\check{x}}\left[\eta_{\check{\boldsymbol{\alpha}}}\right] = \mathbb{P}_{\check{x}}\left(\sigma_{\check{\boldsymbol{\alpha}}} < \infty\right) \mathbb{E}_{\check{\boldsymbol{\alpha}}}\left[\eta_{\check{\boldsymbol{\alpha}}}\right] \quad (6.6)$$

(recall the computation (3.17) in the proof of Theorem 3.14) and $\check{\boldsymbol{\alpha}}$ is accessible, $\mathbb{E}_{\check{\boldsymbol{\alpha}}}[\eta_{\check{\boldsymbol{\alpha}}}] = \infty$, showing that $\check{\boldsymbol{\alpha}}$ is recurrent. On the other hand, if $\check{\boldsymbol{\alpha}}$ is recurrent, then again by (6.6), for all $\check{x} \in C \times \{0,1\}$, $\mathbb{E}_{\check{x}}[\eta_{\check{\boldsymbol{\alpha}}}] = \infty$, implying, again via (6.3), that $\mathbb{E}_{\delta_x^*}[\eta_{\check{\boldsymbol{\alpha}}}] = \infty$ for all $x \in C$. Thus, by (6.5), C is recurrent. □

General phi-irreducible chains and invariant measures

Theorem 6.7. *Let P be a recurrent phi-irreducible transition kernel that admits an accessible $(1, \varepsilon, \nu)$-small set $C \in \mathcal{X}$. Then it admits a non-trivial invariant measure π such that any other nontrivial P-invariant measure μ with $\mu(C) < \infty$ is proportional to π. In addition, any such invariant measure is a maximal irreducibility measure.*

Proof. We start off the proof with the following exercise.

Exercise 6.8. *Show that $\mu \in \mathsf{M}_+(\mathcal{X})$ is invariant with respect to P if and only if μ^* is invariant with respect to $\check{P}$.*

Now, given some $\check{P}$-invariant $\check{\mu} \in \mathsf{M}_+(\check{\mathcal{X}})$, define

$$\mu : \mathcal{X} \ni A \mapsto \int_{C \times \{0\}} R(x, A)\, \check{\mu}(\mathrm{d}\check{x}) + \int_{C^{\complement} \times \{0\}} P(x, A)\, \check{\mu}(\mathrm{d}\check{x}) + \check{\mu}(\mathsf{X} \times \{1\})\nu(A). \tag{6.9}$$

Exercise 6.10. *Check that $\check{\mu}\check{P} = \mu^*$.*

On the basis of the previous exercises, the proof will be completed in the exercise class on 19 April.

□

References

[1] S. P. Meyn and R. L. Tweedie. *Markov Chains and Stochastic Stability.* Springer, 1993.