SF3953 Markov Chains and Processes

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Lecture 8
Limit theorems for Markov chains

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Some background: Birkhoff's ergodic theorem

Theorem (Birkhoff's ergodic theorem)

Let $(\Omega, \mathcal{A}, \mathbb{P}, T)$ *be a dynamical system and* Y *a random variable such that* $\mathbb{E}[|Y|] < \infty$. *Then,* $\mathbb{P}$*-a.s.,*

$$\lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} Y \circ T^k = \mathbb{E}[Y \mid \mathcal{J}]. \tag{1}$$

Moreover, the convergence holds in $L^1(\Omega, \mathcal{A}, \mathbb{P})$.

Some background: martingale convergence

The following is a consequence of the classical *martingale convergence theorem*.

Theorem

Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ *and* $\{\mathcal{F}_k : k \in \mathbb{N}\}$ *be a filtration of* $\mathcal{F}$*. Then the sequence* $\{\mathbb{E}[X \mid \mathcal{F}_k] : k \in \mathbb{N}\}$ *converges* $\mathbb{P}$*-a.s. and in* $L^1(\Omega, \mathcal{F}, \mathbb{P})$ *to* $\mathbb{E}[X \mid \mathcal{F}_\infty]$*, where* $\mathcal{F}_\infty = \sigma(\cup_{k=0}^{\infty} \mathcal{F}_k)$*.*

Some background: CLT for martingale difference sequences

Recall that $\{(Z_k, \mathcal{F}_k) : k \in \mathbb{N}\}$ is martingale difference sequence if (i) $\mathbb{E}[|Z_k|] < \infty$ and (ii) $\mathbb{E}[Z_{k+1} \mid \mathcal{F}_k] = 0$ for all $k \in \mathbb{N}$.

Theorem

Let $\{(Z_k, \mathcal{F}_k) : k \in \mathbb{N}\}$ be square integrable martingale difference sequence. Assume that there exists $\sigma > 0$ such that as n tends to infinity,

$$n^{-1} \sum_{k=0}^{n-1} \mathbb{E}\left[Z_{k+1}^2 \mid \mathcal{F}_k\right] \xrightarrow{\mathbb{P}} \sigma^2,$$

$$n^{-1} \sum_{k=0}^{n-1} \mathbb{E}\left[Z_{k+1}^2 \mathbb{1}_{\{|Z_{k+1}|>\delta\sqrt{n}\}} \mid \mathcal{F}_k\right] \xrightarrow{\mathbb{P}} 0,$$

for all $\delta > 0$. Then $n^{-1/2} \sum_{k=0}^{n-1} Z_k \stackrel{\mathbb{P}}{\Longrightarrow} \sigma V$, where V is standard normally distributed.