KTH Mathematics

Tentamen i 5B1575 Finansiella Derivat.
Måndag 27 augusti 2007 kl. 14.00–19.00.

Answers and suggestions for solutions.

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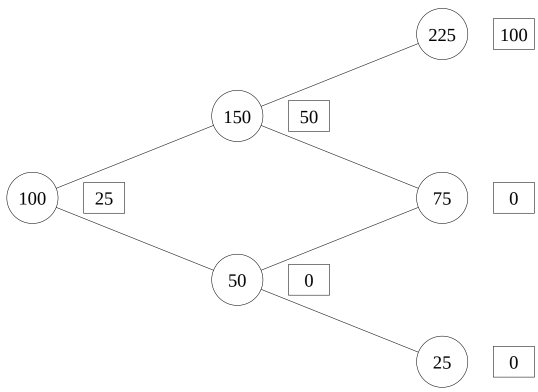
1. (a) For the martingale probabilities we have

$$q = \frac{1+r-d}{u-d} = 0.5$$

Using them we obtain the following binomial tree where the value of the stock is written in the nodes, and the value of the option is written in the adjacent boxes.



The price of the call option is thus 25 kr.

(b) A replicating portfolio is a self-financing portfolio which has the same value as the claim at the exercise date. It is easy to check that the value of a portfolio consisting of K bonds, a long position in a call option with strike price K and exercise date T, and a short position in the underlying stock itself will be equal to the value of the put option at time T. The portfolio is also self-financing since it is constant.

(c) i. A probability measure Q is a risk neutral martingale measure for the standard Black-Scholes model if it

- is equivalent to P, and
- the process S/B is a Q-martingale.

ii. Under the martingale measure Q the dynamics of S are given by

$$dS_t = rS_t dt + \sigma S_t dV_t,$$

where V is a Q-Wiener process. Using Itô's formula on $R = \ln(S_t/S_0)$ we obtain

$$\begin{aligned} dR_t &= \frac{1}{S_t} dS_t - \frac{1}{2}\frac{1}{S_t^2}(dS_t)^2 \\ &= \left(r - \frac{1}{2}\sigma^2\right) dt + \sigma dV_t \end{aligned}$$

Since $R_0 = 0$ we obtain

$$R_t = \left(r - \frac{1}{2}\sigma^2\right) t + \sigma V_t,$$

and thus R_t is normally distributed with expectation $r - \sigma^2/2$ and variance $\sigma^2 t$.

2. (a) The price of the claim at time $t \in [0, T]$ is given by

$$\Pi_t[X] = e^{-r(T-t)} E^Q \left[S_T^\beta \middle| \mathcal{F}_t \right]$$

The dynamics of S under Q are given by

$$dS_t = rS_t dt + \sigma S_t dV_t,$$

where V is a Q-Wiener process. Let $Z = S^\beta$ and use Itô's formula to find the differential of Z

$$\begin{aligned} dZ &= \beta S^{\beta-1} dS + \frac{1}{2}\beta(\beta - 1) S^{\beta-2}(dS)^2 \\ &= \left(\beta r + \frac{1}{2}\beta(\beta - 1)\sigma^2\right) Z dt + \beta\sigma Z dV. \end{aligned}$$

Let $c = \beta r + \beta(\beta-1)\sigma^2/2$. Integrating we obtain

$$Z_u = Z_t + \int_t^u cZ_s ds + \int_t^u \beta\sigma Z_s dV_s.$$

Now take the conditional expectation with respect to $\mathcal{F}_t$

$$E[Z_u|\mathcal{F}_t] = Z_t + \int_t^u cE[Z_s|\mathcal{F}_t]ds + 0.$$

Let $m_u = E[Z_u|\mathcal{F}_t]$ and take derivatives with respect to u

$$\begin{cases} \dot{m} &= cm, \\ m_t &= Z_t. \end{cases}$$

Solving the ODE above we get

$$m_u = Z_t e^{c(u-t)}.$$

The price of the claim is given by $e^{-r(T-t)}m_T$, i.e.

$$\Pi_t[X] = S_t^\beta e^{(c-r)(T-t)}, \quad \text{with } c = \beta r + \frac{1}{2}\beta(\beta-1)\sigma^2.$$

(b) We know that $\Pi(t) = F(t, S_t)$. Therefore using Itô's formula on F we obtain (in the formulas below sub-indices denote partial derivatives, for example $F_s = \frac{\partial F}{\partial s}$)

$$\begin{aligned} dF &= F_t dt + F_s dS + \frac{1}{2}F_{ss}(dS)^2 && (1) \\ &= \left(F_t + \frac{1}{2}\sigma^2 S^2 F_{ss}\right)dt + F_s dS. && (2) \end{aligned}$$

In order for a portfolio $h = (h^B, h^S)$ to be self-financing it must satisfy

$$dV_t^h = h^B dB_t + h^S dS_t.$$

Now comparing with equation (2) we see that we must have

$$h_t^S = F_s(t, S_t) = \beta S_t^{\beta-1} e^{(c-r)(T-t)}, \quad c = \beta r - \frac{1}{2}\beta(\beta-1)\sigma^2.$$

Since $V_t^h = h^B B_t + h^S S_t$ we must have

$$h^B = \frac{F - SF_s}{e^{rt}} = (1-\beta)S_t^\beta e^{(c-r)(T-t)}e^{-rt} \quad c = \beta r - \frac{1}{2}\beta(\beta-1)\sigma^2.$$

You may recognize the continuously rebalanced delta hedge.

3. (a) The price of the option on the underlying stock which pays no dividends will be greater than the price of the option on the dividend paying stock, given that the stock price is the same today. This is because the price of the stock takes future dividend payments into account, but the holder of the option will not benefit from them, and the dividend payments will cause the stock price to fall making it less likely that the option will be exercised.

(b) Just as in the ordinary Black-Scholes model the price of the option is given by

$$\Pi_t = e^{-r(T-t)} E^Q \left[\max\{S_T - K, 0\} \middle| \mathcal{F}_t \right]$$

However, in a model with dividends it is the normalized gains process, G, which should be a martingale under the martingale measure Q. The normalized gains process for the stock is given by

$$G^Z(t) = \frac{S_t}{B_t} + \int_0^t \frac{\delta S_u}{B_u} du.$$

This can be seen to be a martingale if the dynamics of S under Q are given by

$$dS_t = (r - \delta) S_t dt + \sigma S_t dV_t,$$

where V is a Q-Wiener process.

If we rewrite the price in the following way

$$\Pi_t = e^{-\delta(T-t)} \cdot \underbrace{e^{-(r-\delta)(T-t)} E^Q \left[\max\{S_T - K, 0\} \middle| \mathcal{F}_t \right]}_{BS formula}$$

we can as indicated use the Black-Scholes formula on the second part with $r - \delta$ for the interest rate and σ for the volatility. We thus have

$$\begin{aligned} \Pi_t &= e^{-\delta(T-t)} [S_t \Phi(d_1) - e^{-(r-\delta)(T-t)} K \Phi(d_2)] \\ &= e^{-\delta(T-t)} S_t \Phi(d_1) - e^{-r(T-t)} K \Phi(d_2) \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}} \left\{ \ln\left(\frac{S(t)}{K}\right) + \left(r - \delta + \frac{1}{2}\sigma^2\right)(T-t) \right\}, \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

4. (a) The Ho-Lee model possesses an affine term structure, i.e. the zero coupon bond prices are of the form $p(t,T) = e^{A(t,T)-B(t,T)r_t}$, where A and B are deterministic functions. To completely specify the bond prices we need to find expressions for A and B. The Itô formula applied to $p(t,T) = e^{A(t,T)-B(t,T)r_t}$ yields

$$\begin{aligned} dp^T &= (A_t - B_t r)p^T dt - Bp^T dr + \frac{1}{2}B^2 p^T (dr)^2 \\ &= \left[A_t - B_t r - \theta B + \frac{1}{2}B^2\rho^2\right] p^T dt - \rho B p^T dU \\ &= \left[A_t - \theta B + \frac{1}{2}\rho^2 B^2 - B_t r\right] p^T dt - \rho B p^T dU. \end{aligned}$$

Under Q we know that $p(t,T)/B(t)$ is a martingale, which means that $p(t,T)$ has to have local return equal to the short rate r. Thus, for $t \geq 0$, and $r \in (-\infty, \infty)$ the following equality has to hold

$$\left[A_t - \theta B + \frac{1}{2}\rho^2 B^2\right] - [B_t + 1]r = 0$$

The only way this is possible is if both square brackets equal zero, this will give you the ordinary differential equations solved by A, and B. The boundary conditions are obtained from the condition that $p(T,T) = 1$. To sum up we have

$$\begin{cases} B_t(t,T) &= -1, \\ B(T,T) &= 0, \end{cases}$$

and

$$\begin{cases} A_t(t,T) &= \theta B(t,T) - \frac{1}{2}\rho^2 B^2(t,T), \\ A(T,T) &= 0. \end{cases}$$

why

$$\begin{aligned} B(t,T) &= (T-t), \\ A(t,T) &= \int_t^T \theta(s)(s-T)ds + \frac{\rho^2}{2}\frac{(T-t)^3}{3}. \end{aligned}$$

(b) Using the T-forward measure Q^T we have the following pricing formula

$$\Pi(t) = p(t,T)E^T\left[\left.\frac{\max\{S(T)-K,0\}}{p(T,T)}\right| \mathcal{F}_t\right],$$

where the super index T indicates that the expectation should be taken under the forward measure Q^T. This can be written as

$$\Pi(t) = p(t,T)E^T\left[\max\{Z(T)-K,0\}|\,\mathcal{F}_t\right].$$

where $Z(t) = S(t)/p(t,T)$. In order to use the extended version of the Black-Scholes formula we need an interest rate and a volatility. Under Q^T Z will be a martingale (it is a price process normalized by the numeraire $p(t,T)$), which means that the drift of Z is zero, and therefore also the interest rate used should be zero. To obtain the volatility first note that Itô applied to $p(t,T) = e^{A(t,T)-B(t,T)r_t}$ yields

$$dp(t,T) = rp(t,T)dt - \rho B(t,T)p(t,T)dU.$$

Then use Itô once more to obtain

$$\begin{aligned} dZ_t &= \left(\frac{S_t}{p(t,T)}\right) = (\ldots)dt + Z_t[\sigma dW_t + \rho B(t,T)dU_t] \\ &= \sqrt{\sigma^2+\rho^2B^2(t,T)}dV_t \end{aligned}$$

From this we see that the volatility of Z is deterministic. Let $\gamma(t) = \sqrt{\sigma^2+\rho^2B^2(t,T)}$. Then under Q^T we have

$$dZ_t = \gamma(t)Z_t dV_t^T$$

for a Q^T-Wiener process V^T.

Applying the extended version of the Black-Scholes formula now yields

$$\begin{aligned} \Pi(t) &= p(t,T)E^T\left[\max\{Z(T)-K,0\}|\,\mathcal{F}_t\right] \\ &= p(t,T)\left[Z(t)\Phi(d_1) - K\Phi(d_2)\right] \\ &= S(t)\Phi(d_1) - Kp(t,T)\Phi(d_2) \end{aligned}$$

where

$$d_1 = \frac{\ln\left(\frac{S(t)}{Kp(t,T)}\right) + \frac{1}{2}\int_t^T \gamma^2(s)ds}{\sqrt{\int_t^T \gamma^2(s)ds}}, \quad d_2 = d_1 - \sqrt{\int_t^T \gamma^2(s)ds}.$$

5. Recall that

$$f(t,T) = Z_t + H(T-t), \tag{3}$$

and that

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t. \tag{4}$$

By definition, we have that $r_t = f(t,t)$, so

(a)
$$r_t = f(t,t) = Z_t + H(0) = Z_t.$$

(b) Using (3), the relation $r_t = Z_t$, the dynamics of r in (4), and the Itô formula we obtain

$$df(t,T) = [\mu(t, r_t) - h(T-t)]dt + \sigma(t, r_t)dW_t$$

where $h(x) = H'(x)$. The HJM drift condition now reads

$$\mu(t, r_t) - h(T-t) = \sigma(t, r_t) \int_t^T \sigma(t, r_t)ds$$

i.e. we obtain the following identity for all t, T, and r

$$\mu(t, r) - h(T-t) = \sigma^2(t, r)(T-t).$$

Denoting $T - t$ by x this reads

$$\mu(t, r) - h(x) = \sigma^2(t, r)x.$$

Setting $x = 0$ yields

$$\mu(t, r) = c$$

where $c = h(0)$. We thus have

$$c - h(x) = \sigma^2(t, r)x$$

which implies that also σ has to be a constant as a function of (t,r). This means that we have the following degenerate form of the Ho-Lee model

$$dr_t = cdt + \sigma dW_t.$$

For fun we note that

$$h(x) = c - \sigma^2 x.$$

Integrating we obtain

$$H(x) = -\frac{\sigma^2}{2}x^2 - cx + d.$$