KTH Mathematics

Tentamen i 5B1575 Finansiella Derivat.
Tisdag 22 maj 2007 kl. 14.00–19.00.

Examinator: Camilla Landén, tel 790 8466.

Tillåtna hjälpmedel: Inga.

Allmänna anvisningar: Lösningarna skall vara lättläsliga och **välmotiverade**. All införd notation skall vara förklarad. Problem rörande integrabilitet behöver ej redas ut.

OBS! Personnummer skall anges på försättsbladet. Endast en uppgift på varje blad. Numrera sidorna och skriv namn på varje blad!

25 poäng inklusive bonuspoäng ger säkert godkänt.

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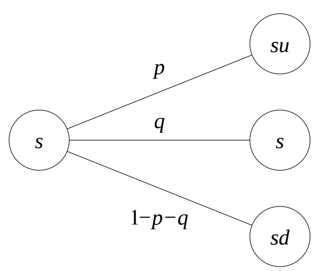
1. (a) Consider a one period model very similar to the one period binomial model, the only difference being that the stock price S can also stay the same with a certain probability q, as depicted in the figure below.



Given that $u = 1.5$, $d = 0.5$ and r=10% is the model free of arbitrage or not? A proof or counterexample is required (you may use the first and second fundamental theorem without proving them). . . . . (3p)

(b) In his thesis from 1900 Louis Bachelier suggested the following model for the (discounted) stock price under the martingale measure Q

$$dS_t = S_0 \sigma dV_t.$$

Here S_0 is the initial value of the stock, σ is a constant, and V denotes a Q-Wiener process. Suppose that the interest rate is zero so that we do not have to worry about discounting.

For $t = 0$ (to avoid conditional expectations) derive the following option pricing formula for a European call option on the stock with price process S, strike price K, and exercise date T

$$C_{Bach}(t) =$$

$$(S_t - K)\Phi\left(\frac{S_t - K}{\sigma S_0 \sqrt{T-t}}\right) + \sigma S_0 \sqrt{T-t}\,\phi\left(\frac{S_t - K}{\sigma S_0 \sqrt{T-t}}\right). \quad (1)$$

Here Φ and ϕ denote the standard normal distribution and density function, respectively, i.e.

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \quad \text{and} \quad \Phi(x) = \int_{-\infty}^{x} \phi(u) du.$$

.... (3p)

Hint: You may, without proving it, use the following result about normal distributions: If $X \in N(\mu, \sigma^2)$ then

$$E[X I_{\{l \le X \le h\}}] =$$

$$= \mu \left[\Phi\left(\frac{h-\mu}{\sigma}\right) - \Phi\left(\frac{l-\mu}{\sigma}\right)\right] + \sigma \left[\phi\left(\frac{l-\mu}{\sigma}\right) - \phi\left(\frac{h-\mu}{\sigma}\right)\right]$$

where I_A denotes the indicator function of the set A.

(c) In 1900 Louis Bachelier presented the formula (1) for the price of a European call option on a stock with price process S, strike price K, and exercise date T.

Compute the Δ of a call option given that it is priced using the formula of Bachelier, and explain how it can be used for hedging if you have sold the option and intend to use the underlying itself for hedging. .... (4p)

2. Consider a standard Black-Scholes market, i.e., a market consisting of a risk free asset, B, with P-dynamics given by

$$\begin{cases} dB_t &= rB_t dt, \\ B_0 &= 1, \end{cases}$$

and a stock, S, with P-dynamics given by

$$\begin{cases} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ S_0 &= s_0. \end{cases}$$

Here W denotes a P-Wiener process and r, α, and σ are assumed to be constants.

(a) Recall that a forward contract on S_T contracted at time t, with time of delivery T, and with forward price $f(t;T,S_T)$ can be seen as a contingent T-claim X with payoff

$$X = S_T - f(t;T,S_T).$$

The forward price is determined at time t in such a way that the price of X is zero at time t, i.e. $\Pi(t;X) = 0$.

Compute the forward price $f(t;T,S_T)$ in the Black-Scholes model. .... (3p)

(b) A *range forward* is a T-claim with payoff X defined by

$$X = \max\{\min\{S_T, K_2\}, K_1\} - f(0;T,S_T),$$

where K_1 and K_2 are constants such that $K_1 < f(0;T,S_T) < K_2$ and, as before, $f(0;T,S_T)$ is the forward price of the stock for settlement at time T.

Compute the arbitrage free price of the range forward at time t for $0 \le t \le T$. .... (3p)

(c) Show that the range forward can be replicated by a portfolio consisting of forwards, and put and call options. .... (4p)

3. In this exercise you will be asked to compute zero coupon bond prices in a two-factor interest rate model. Suppose that the short rate has the form

$$r(t) = X_1(t) + X_2(t),$$

where the processes X_1 and X_2 solve

$$dX_i(t) = \kappa_i[\theta_i - X_i(t)]dt + \sigma_i dW_i(t), \quad i = 1, 2. \tag{2}$$

Here κ_i, θ_i, and σ_i are constants for i=1,2, and W_1 and W_2 are independent Q-Wiener processes.

(a) Using that the solution to the SDE (2) is given by

$$X_i(T) = e^{-\kappa_i(T-t)}X_i(t) + \kappa_i\theta_i \int_t^T e^{-\kappa_i(T-s)}ds + \sigma_i \int_t^T e^{-\kappa_i(T-s)}dW_i(s),$$

show that zero coupon bond prices in this model are of the form

$$p(t,T) = e^{A(t,T)-B(t,T)X_1(t)-C(t,T)X_2(t)},$$

where A, B, and C are deterministic functions. .... (4p)

(b) Derive ordinary differential equations for A, B, and C. Do not forget the boundary conditions. (You do not have to solve the equations.) .... (6p)

Hint: Use that you know something about the drift of the bond prices under Q, and that you can also compute the drift using the Itô formula.

4. Suppose you wish to price a cross-currency swap. To do so suppose that the domestic risk free asset B_d, the foreign risk free asset B_f, the domestic zero coupon bond prices $p_d(\cdot, T)$, $T \geq 0$, the foreign zero coupon bond prices $p_f(\cdot, T)$, $T \geq 0$, and the exchange rate X, which is used to convert foreign payoffs into domestic currency, have the following dynamics under the domestic martingale measure Q^d

$$\begin{aligned}
dB_d(t) &= r_d(t)B_d(t)dt, \\
dB_f(t) &= r_f(t)B_f(t)dt, \\
dX(t) &= [r_d(t) - r_f(t)]X(t)dt + \sigma(t)X(t)dW(t), \\
dp_d(t,T) &= r_d(t)p_d(t,T)dt + \nu_d(t,T)p_d(t,T)dW(t), \\
dp_f(t,T) &= [r_f(t) - \sigma(t)\nu_f(t,T)]p_f(t,T)dt + \nu_f(t,T)p_f(t,T)dW(t).
\end{aligned} \tag{3}$$

Here ν_d, ν_f, and σ are deterministic functions, and W is a standard Q^d-Wiener process. The short rates r_d and r_f are stochastic.

Now, fix T and U such that $T < U$. Pricing a cross-currency swap basically boils down to pricing a U-claim Z with the following payoff in domestic currency

$$Z = \frac{1}{p_f(T,U)} - \frac{1}{p_d(T,U)}.$$

You will be asked to look at the pricing in steps.

(a) Compute the domestic arbitrage free price of a (T,U) *roll bond*, i.e. a claim with the following payoff in domestic currency at time U

$$Y_{roll} = \frac{1}{p_d(T,U)}.$$

The expression for the price may contain any of the processes specified in equation (3), as they are considered as given. . . (4p)
Hint: This can be done by finding a replicating portfolio for the payoff (or by computing an expectation).

(b) A (T,U) *quanto roll bond* is a claim with the following payoff in **domestic** currency at time U

$$Y_{q-roll} = \frac{1}{p_f(T,U)}.$$

Compute the domestic arbitrage free price of a (T,U) *quanto roll bond* for $t \geq T$. Again, the expression for the price may contain any of the processes specified in equation (3), as they are considered as given. . . . . (2p)

(c) Let

$$V(t;T,U) = \frac{p_d(t,U)p_f(t,T)G(t;T,U)}{p_f(t,U)}, \quad \text{for } t \in [0,T]$$

where

$$G(t;T,U) = \exp\left\{\int_t^T g(u;T,U)du\right\}$$

and

$$g(t;T,U) = [\nu_f(t,U) - \nu_f(t,T)] \cdot [\sigma(t) + \nu_f(t,U) - \nu_d(t,U)].$$

This means that

$$\begin{aligned} \frac{dV(t;T,U)}{V(t;T,U)} &= \frac{dp_d(t,U)}{p_d(t,U)} - \frac{dp_f(t,U)}{p_f(t,U)} + \frac{dp_f(t,T)}{p_f(t,T)} \\ &\quad + [\nu_f(t,T) - \nu_f(t,U)]\sigma(t)dt. \end{aligned}$$

Define the portfolio $h = (h_1, h_2, h_3)$ by

$$h_1(t) = \frac{V(t;T,U)}{p_d(t,U)}, \quad h_2(t) = -\frac{V(t;T,U)}{\tilde{p}_f(t,U)}, \quad h_3(t) = \frac{V(t;T,U)}{\tilde{p}_f(t,T)},$$

where $\tilde{p}_f(t,S) = X(t)p_f(t,S)$, i.e. the price of a foreign bond converted into domestic currency.

Suppose that you trade on the domestic market. Show that if you hold h_1 domestic U-bonds, h_2 foreign U-bonds (in the domestic market you will then actually have a position h_2 in $\tilde{p}_f(\cdot, U)$), and h_3 foreign T-bonds (h_3 position in $\tilde{p}_f(\cdot, T)$) this replicates the T-claim χ with a payoff in domestic currency given by

$$\chi = \frac{p_d(T,U)}{p_f(T,U)}.$$

.... (4p)

5. Consider a standard Black-Scholes market described in detail in exercise 2. Furthermore consider a contingent T-claim X with payoff

$$X = \max\left\{\frac{1}{T}\int_0^T S(u)du - S(T), 0\right\}.$$

The claim X can thus be seen as a European put option for which the strike price is determined by the arithmetic mean of the stock price.

Let $\Pi(t)$ denote the price of X at time t and define

$$Z(t) = \frac{\frac{1}{T}\int_0^t S(u)du}{S(t)} \qquad \text{for } t \leq T.$$

Show that

$$\frac{\Pi(t)}{S(t)} = F(t, Z(t)),$$

where the function F solves the following PDE

$$\begin{cases} F_t(t,z) + \left(\frac{1}{T} - rz\right) F_z(t,z) + \frac{1}{2}\sigma^2 z^2 F_{zz}(t,z) = 0, \\ F(T,z) = \max\{z-1, 0\}. \end{cases}$$

.... (10p)

Hint: Change numeraires and use Feynman-Kač.

Hints:
You are free to use the following in any of the above exercises.

- The density function of a normally distributed random variable with expectation m and variance σ^2 is given by

$$\phi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/(2\sigma^2)}.$$

- Let Φ denote the cumulative distribution function for the N(0, 1) distribution. Then

$$\Phi(-x) = 1 - \Phi(x).$$

- The standard Black-Scholes formula for the price $\Pi(t)$ of a European call option with strike price K and time of maturity T is $\Pi(t) = F(t, S(t))$, where

$$F(t,s) = s\Phi[d_1(t,s)] - e^{-r(T-t)}K\Phi[d_2(t,s)].$$

  Here Φ is the cumulative distribution function for the $N(0,1)$ distribution and

$$\begin{aligned} d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}, \\ d_2(t,s) &= d_1(t,s) - \sigma\sqrt{T-t}. \end{aligned}$$

- Parameterise the Vasiček model in the following way under the risk neutral martingale measure Q

$$dr(t) = (b - ar)dt + \sigma dV. \tag{4}$$

  Proposition 0.1 (The Vasiček term structure) *In the Vasiček model, parameterised as in (4) under Q, the zero-coupon bond prices are given by*

$$p(t,T) = e^{A(t,T) - B(t,T)r(t)}, \tag{5}$$

  where

$$B(t,T) = \frac{1}{a}\left\{1 - e^{-a(T-t)}\right\}, \tag{6}$$

$$A(t,T) = \frac{[B(t,T) - T + t](ab - \frac{1}{2}\sigma^2)}{a^2} - \frac{\sigma^2 B^2(t,T)}{4a}. \tag{7}$$