KTH Mathematics

Exam in SF2975 Financial Derivatives.
Tuesday May 20 2008 14.00-19.00.

Answers and suggestions for solutions.

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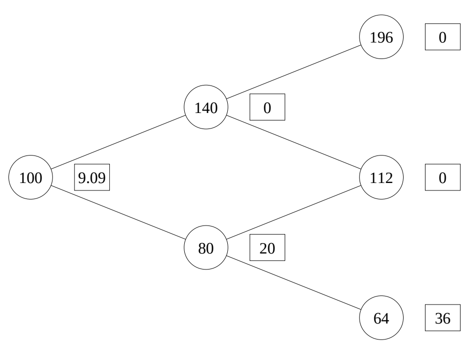
1. (a) For the martingale probabilities we have

$$q = \frac{1+r-d}{u-d} = 0.5$$

Using them we obtain the following binomial tree where the value of the stock is written in the nodes, and the value of the option is written in the adjacent boxes. The value 20 adjacent to the node with stock price 80 is obtained as $\max\{100-80, \frac{1}{1.1}(0.5\cdot 0 + 0.5\cdot 36)\}$. Thus, an early exercise of the option is optimal at this node!



The price of the put option is thus 9.09 kr.

(b) Fix an arbitrary contingent claim $\phi(Z)$, where Z denotes a random variable which equals u with probability p and d with probability $1-p$. To show that the market is complete we have to show that we can find a replicating portfolio $\mathbf{h} = (x, y)$ for this contract. This means solving the following set of equations

$$\begin{cases} x(1+r) + ysu &= \phi(u), \\ x(1+r) + ysd &= \phi(d), \end{cases}$$

This is a system of linear equations in x and y, which has a unique solution if

$$\det\begin{pmatrix} 1+r & su \\ 1+r & sd \end{pmatrix} \neq 0$$

that is if $u \neq d$. Since $u > d$ this is the case, and the solution is given by

$$\begin{aligned} x &= \frac{1}{1+r} \cdot \frac{u\phi(d) - d\phi(u)}{u-d}, \\ y &= \frac{1}{s} \cdot \frac{\phi(u) - \phi(d)}{u-d}. \end{aligned}$$

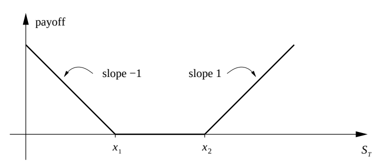
Since every contingent claim can be replicated the market is complete.

(c) i. A replication portfolio for the contingent T-claim X is a self-financing portfolio h, such that

$$V^h(T) = X \quad P - a.s.$$

for the corresponding value process V^h.

ii. Since the portfolio is a constant portfolio it is self-financing and therefore it is a replicating portfolio for some claim. Some thought reveals that the payoff functions is the following:



2. The terminal payoff of a forward-start call option is given by the stochastic variable X defined by

$$X = \max\{S_T - S_{T_0}, 0\}.$$

The price of a forward-start call option at time $t \in [0, T_0]$ is given by

$$\begin{aligned} \Pi_t[X] &= e^{-r(T-t)} E^Q \left[\max\{S_T - S_{T_0}, 0\} \middle| \mathcal{F}_t \right] \\ &= e^{-r(T-t)} E^Q \left[E^Q \left[\max\{S_T - S_{T_0}, 0\} \middle| \mathcal{F}_{T_0} \right] \middle| \mathcal{F}_t \right] \end{aligned}$$

Now use that $S_T = S_{T_0} \exp\left\{ \left(r - \frac{1}{2}\sigma^2 \right)(T - T_0) + \sigma(V_T - V_{T_0}) \right\}$.

$$\begin{aligned} \Pi_t[X] &= e^{-r(T-t)} E^Q \left[E^Q \left[S_{T_0} \max\left\{ e^{\left(r - \frac{1}{2}\sigma^2\right)(T-T_0) + \sigma(V_T - V_{T_0})} - 1, 0 \right\} \middle| \mathcal{F}_{T_0} \right] \middle| \mathcal{F}_t \right] \\ &= e^{-r(T-t)} E^Q \left[\frac{S_{T_0}}{e^{-r(T-T_0)}} \times \right. \\ &\quad \left. \times e^{-r(T-T_0)} E^Q \left[\max\left\{ e^{\left(r - \frac{1}{2}\sigma^2\right)(T-T_0) + \sigma(V_T - V_{T_0})} - 1, 0 \right\} \middle| \mathcal{F}_{T_0} \right] \middle| \mathcal{F}_t \right] \\ &= e^{rt} E^Q \left[\frac{S_{T_0}}{e^{rT_0}} c(T_0, 1, 1, T, r, \sigma) \middle| \mathcal{F}_t \right] \\ &= S_t c(T_0, 1, 1, T, r, \sigma). \end{aligned}$$

For the last equality we used the fact that S_t / B_t is a Q-martingale. Here $c(t, s, K, T, r, \sigma)$ denotes the standard Black-Scholes price at time t of a European call option with exercise price K and expiry date T, when the current price of the underlying is s, the interest rate is r, and the volatility of the underlying is σ.

3. (a) A short rate model is said to have an affine term structure if zero coupon bond prices can be written on the following form

$$p(t,T) = e^{A(t,T)-B(t,T)r_t},$$

where A and B are deterministic functions.

Sufficient conditions on μ and σ which guarantee the existence of an affine term structure are that μ and σ^2 are affine in r (and that there exists solutions to two certain ordinary differential equation, see below), i.e.

$$\begin{cases} \mu(t,r) &= a(t)r + b(t), \\ \sigma^2(t,r) &= c(t)r + d(t). \end{cases}$$

To see this insert these expression into the term structure equation

$$\begin{cases} F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - rF^T = 0, \\ F(T,r) = 1. \end{cases}$$

(Here $F^T(t,r_t) = p(t,T)$, and we have used the notation $F_t^T = \partial F^T/\partial t$, etc.) This will after some rewriting give you

$$\begin{cases} A_t(t,T) - b(t)B(t,T) + \frac{1}{2}d(t)B^2(t,T) \\ + \Big(1 + B_t(t,T) + a(t)B(t,T) - \frac{1}{2}c(t)B^2(t,T)\Big)\, r = 0, \\ e^{A(T,T)-B(T,T)r} = 0. \end{cases}$$

This equation should hold for all t and r, so there will be an affine term structure if A and B solve the following ordinary differential equations

$$\begin{cases} A_t(t,T) &= b(t)B(t,T) - \frac{1}{2}d(t)B^2(t,T), \\ A(T,T) &= 0, \end{cases} \tag{1}$$

and

$$\begin{cases} B_t(t,T) + a(t)B(t,T) - \dfrac{1}{2}c(t)B^2(t,T) &= -1, \\ B(T,T) &= 0. \end{cases} \tag{2}$$

(b) Since the short rate solves a linear SDE it possesses an affine term structure, thus $p(t,T) = e^{A(t,T)-B(t,T)r_t}$. This means that the yield is

$$y(t,T) = -\frac{\ln p(t,T)}{T-t} = -\frac{A(t,T)}{T-t} + \frac{B(t,T)}{T-t} =: a(t,T) + b(t,T)r_t.$$

Therefore the correlation is

$$Corr(y(t,T_1), y(t,T_2)) = Corr(b(t,T_1)r_t, b(t,T_2)r_t) = 1.$$

This means that a shock to the interest rate curve at time t is transmitted equally through all maturities, which seems unlikely. This is the reason multi-factor models have been introduced (or at least one of the reasons for doing so).

(c) The numeraire asset under Q^T is the T-bond with price process $p(t,T)$.

The Girsanov kernel for transformation between Q and Q^T is the volatility of the T-bond (the new numeraire asset). Use Itô's formula on $p(t,T) = e^{A(t,T)-B(t,T)r_t}$ to obtain

$$dp(t,T) = \ldots dt - \sigma(t,r_t)B(t,T)p(t,T)dV_t.$$

The volatility is thus

$$\nu(t,T) = -\sigma(t,r_t)B(t,T),$$

where the function B is the solution to equation (2).

4. The price of the pay later option is given by

$$\Pi_t = e^{-r_d(T-t)}E^d\left[(X_T - K - p)I\{X_T > K\}\middle|\,\mathcal{F}_t\right]$$

where the super-index d on the expectation indicates that the expectation should be taken under the domestic martingale measure Q^d. Rewriting this a bit we have

$$\Pi_t = e^{-r_d(T-t)}E^d\left[(X_T - K)I\{X_T > K\}\middle|\,\mathcal{F}_t\right] - pE^d\left[I\{X_T > K\}\middle|\,\mathcal{F}_t\right]$$

Recall that the dynamics of X under Q^d are given by

$$dX = (r_d - r_f)X dt + \sigma_X X dW^d,$$

where W^d is a Q^d-Wiener process. (This follows from the fact that $\widetilde{B}^f/B^d = XB^f/B^d$ is a Q^d-martingale.) Some further rewriting will put us in a position where we can use Black-Scholes formula for the first term.

$$\begin{aligned}
\Pi_t &= e^{-r_f(T-t)}e^{-(r_d-r_f)(T-t)}E^d\left[(X_T - K)I\{X_T > K\}\middle|\,\mathcal{F}_t\right] - e^{-r_d(T-t)}pQ^d(X_T > K) \\
&= e^{-r_f(T-t)}\left\{X_t N[d_1(t)] - e^{-(r_d-r_f)(T-t)}KN[d_2(t)]\right\} - e^{-r_d(T-t)}pQ^d(X_T > K),
\end{aligned}$$

where

$$\begin{aligned}
d_1(t) &= \frac{1}{\sigma_X\sqrt{T-t}}\left\{\ln\left(\frac{X_t}{K}\right) + \left(r_d - r_f + \frac{1}{2}\sigma_X^2\right)(T-t)\right\}, \\
d_2(t) &= d_1 - \sigma_X\sqrt{T-t}.
\end{aligned}$$

Since X is a GBM it has the explicit solution

$$X_T = X_t\exp\left\{\left(r_d - r_f - \frac{1}{2}\sigma_X^2\right)(T-t) + \sigma_X(W_T^d - W_t^d)\right\},$$

and we see that $X_T = X_t e^Z$, where $Z \in N((r_d - r_f - \sigma_X^2/2)(T-t), \sigma_X^2(T-t))$. The probability in the second term is now easily computed to be

$$Q^d(X_T > K) = N[d_2(t)].$$

The price of the pay later option at time t is thus

$$\Pi_t = e^{-r_f(T-t)} X_t N[d_1(t)] - e^{-r_d(T-t)} K N[d_2(t)] - e^{-r_d(T-t)} p N[d_2(t)],$$

and the correct premium p is therefore given by

$$p = e^{(r_d - r_f)T} X_0 \frac{N[d_1(0)]}{N[d_2(0)]} - K.$$

5. (a) The value of the portfolio is

$$V_t = X_t S_t + 1 \cdot A_t. \tag{3}$$

(b) For a self-financing portfolio we should have $dV^h = h dG$, where G is the gain process, so

$$dV_t = X_t dS_t + 1 \cdot [dA_t + dD_t]. \tag{4}$$

(c) We get that

$$X_t = \frac{V_t - A_t}{S_t}.$$

Applying the Itô formula to this and using (3) and (4) we obtain

$$\begin{aligned}
dX &= d\left(\frac{V}{S}\right) - d\left(\frac{A}{S}\right) \\
&= \frac{1}{S} dV - \frac{V}{S^2} dS - \frac{1}{S^2} dV dS + \frac{1}{2}\frac{2V}{S^3}(dS)^2 \\
&\quad - \left\{\frac{1}{S} dA - \frac{A}{S^2} dS - \frac{1}{S^2} dA dS + \frac{1}{2}\frac{2A}{S^3}(dS)^2\right\} \\
&= \frac{1}{S}(dA + dD + X dS) - \frac{A + XS}{S^2} dS \\
&\quad - \frac{1}{S^2}(dA + dD + X dS) dS + \frac{A + XS}{S^3}(dS)^2 \\
&\quad - \frac{1}{S} dA + \frac{A}{S^2} dS + \frac{1}{S^2} dA dS - \frac{A}{S^3}(dS)^2 \\
&= \frac{1}{S} dD - \frac{1}{S^2} dD dS
\end{aligned}$$

Finally, using that $X(0) = 0$ we get

$$X_t = \int_0^t \frac{1}{S_u} dD_u - \int_0^t \frac{1}{S_u^2} dS_u dD_u.$$

(d) Under Q^S all price processes of non-dividend paying assets normalized by S should be martingales. Since V represents a non-dividend paying asset the following process should be a Q^S-martingale

$$\frac{V_t}{S_t} = \frac{X_t S_t + A_t}{S_t} = \frac{A_t}{S_t} + X_t = \frac{A_t}{S_t} + \int_0^t \frac{1}{S_u} dD_u - \int_0^t \frac{1}{S_u^2} dD_u dS_u$$