KTH Mathematics

Exam in SF2975 Financial Derivatives.
Monday June 2 2008 14.00-19.00.

Answers and suggestions for solutions.

---

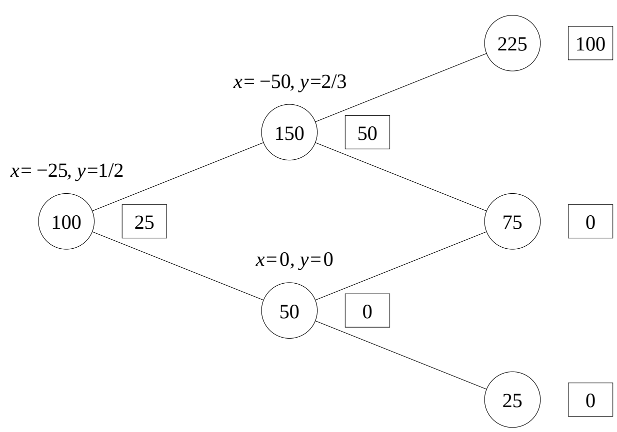
1. (a) To obtain the replicating portfolio at $t = 0$ we have to solve the following set of equations

$$\begin{cases} x + y \cdot 150 = 50, \\ x + y \cdot 50 = 0, \end{cases}$$

since regardless of whether the stock price goes up or down the value of the portfolio should equal the value of the option. This yields

$$x = -25, \qquad y = \frac{1}{2}.$$

Using the same method we find the rest of the replicating portfolio strategy and it is shown in the figure below.



That the portfolio strategy is self-financing is seen from the following equations

$$\begin{cases} -25 + \frac{1}{2} \cdot 150 = -50 + \frac{2}{3} \cdot 150, \\ -25 + \frac{1}{2} \cdot 50 = 0 + 0 \cdot 50, \end{cases}$$

(b) i. The arbitrage bounds for the interest rate r are

$$0.5 \leq (1 + r) \leq 1.5.$$

ii. Both the price of stock and the price of the option have to satisfy the risk-neutral valuation principle. This gives us the following set of equations

$$\begin{cases} 100 &= \frac{1}{1+r}[q \cdot 150 + (1-q) \cdot 50], \\ 22 &= \frac{1}{1+r}[q \cdot 42 + (1-q) \cdot 0]. \end{cases}$$

Solving these equations we find that r=5% (and q=0.55).

(c) i. We have that

$$V_t(h) = h_t^B B_t + h_t^S S_t = B_t + S_t$$

Applying Itô's formula to this we get

$$\begin{aligned} dV_t(h) &= dB_t + dS_t \\ &\neq \tfrac{S_t}{B_t} dB_t + \tfrac{B_t}{S_t} dS_t. \end{aligned}$$

The portfolio defined by $\mathbf{h}_t = \left(\frac{S_t}{B_t}, \frac{B_t}{S_t}\right)$ is therefore **not** self-financing.

ii. Using Itô's formula and the fact that

$$dS_t = rS_t dt + \sigma S_t dV_t$$

under the martingale measure Q (V denotes a Q-Wiener process) we obtain

$$\begin{aligned} dX_t &= -\beta S_t^{-\beta-1} dS_t + \frac{1}{2}(-\beta)(-\beta-1) S_t^{-\beta-2}(dS_t)^2 \\ &= -\beta r S_t^{-\beta} dt - \beta\sigma S_t^{-\beta} dV_t + \frac{1}{2}(\beta+\beta^2)\sigma^2 S_t^{-\beta} dt \\ &= \{ \text{ insert that } \beta = 2r/\sigma^2 \ \} \\ &= rX_t dt - \beta\sigma X_t dV_t \end{aligned}$$

Since the process has a local rate of return of r under the martingale measure Q it represents a tradable asset.

2. (a) If we note that

$$\max\{S_T, K\} = K + \max\{S_T - K, 0\}$$

we can write down the price Π of the option as

$$\Pi_t = e^{-r(T-t)} E^Q[K + \max\{S_T - K, 0\} | \mathcal{F}_t] = e^{-r(T-t)} K + c(t, S_t, K, T, r, \sigma).$$

Here $c(t, s, K, T, r, \sigma)$ denotes the standard Black-Scholes price at time t of a European call option with exercise price K and expiry date T, when the current price of the underlying is s, the interest rate is r, and the volatility of the underlying is σ. Using the hints on the last page of the exam we obtain

$$\Pi_t = e^{-r(T-t)} K + S_t N[d_1(t, S_t)] - e^{-r(T-t)} K N[d_2(t, S_t)].$$

Finally using that $N(-x) = 1 - N(x)$ we get the following price

$$\Pi_t = e^{-r(T-t)} K N[-d_2(t, S_t)] + S_t N[d_1(t, S_t)].$$

(b) The price of the option at time $t \in [0, T_0]$ is given by

$$\begin{aligned} \Pi_t[X] &= e^{-r(T-t)} E^Q \left[\max\{S_T, S_{T_0}\} \middle| \mathcal{F}_t \right] \\ &= e^{-r(T-t)} E^Q \left[E^Q \left[\max\{S_T, S_{T_0}\} \middle| \mathcal{F}_{T_0} \right] \middle| \mathcal{F}_t \right] \end{aligned}$$

Now use that $S_T = S_{T_0} \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T - T_0) + \sigma(V_T - V_{T_0})\right\}$.

$$\begin{aligned}
\Pi_t[X] &= e^{-r(T-t)} E^Q\left[E^Q\left[S_{T_0} \max\left\{ e^{\left(r-\frac{1}{2}\sigma^2\right)(T-T_0)+\sigma(V_T-V_{T_0})}, 1\right\} \middle| \mathcal{F}_{T_0}\right] \middle| \mathcal{F}_t\right] \\
&= e^{-r(T-t)} E^Q\left[\frac{S_{T_0}}{e^{-r(T-T_0)}} \times \right. \\
&\quad \left. \times e^{-r(T-T_0)} E^Q\left[\max\left\{ e^{\left(r-\frac{1}{2}\sigma^2\right)(T-T_0)+\sigma(V_T-V_{T_0})}, 1\right\} \middle| \mathcal{F}_{T_0}\right] \middle| \mathcal{F}_t\right] \\
&= e^{rt} E^Q\left[\frac{S_{T_0}}{e^{rT_0}} \Pi^a(T_0, 1, 1, T, r, \sigma) \middle| \mathcal{F}_t\right] \\
&= S_t \Pi^a(T_0, 1, 1, T, r, \sigma) = S_t(e^{-r(T-T_0)} N[-d_2(t,1)] + N[d_1(t,1)]).
\end{aligned}$$

For the second last equality we used the fact that S_t/B_t is a Q-martingale. $\Pi^a(t, s, K, T, r, \sigma)$ denotes the price at time t of the option in exercise (a), when the current price of the underlying is s, the interest rate is r, and the volatility of the underlying is σ.

3. One way to show that there is arbitrage in the model is to show that there does not exist a martingale measure for the model. Since the filtration is the natural filtration generated by the Wiener process W, we only have to consider Girsanov transformations. Define a Girsanov transformation by

$$d\tilde{Q} = L(t)dP, \qquad \text{on } \mathcal{F}_t,$$

where

$$\begin{cases} dL_t &= L_t g_t dW_t, \\ L_0 &= 1. \end{cases}$$

From Girsanov's theorem we have that

$$dW_t = g_t dt + dV_t,$$

where V is a $\tilde{Q}$-Wiener process. Thus, the $\tilde{Q}$-dynamics of X and Y are given by

$$\begin{cases} dX_t &= (\alpha + \sigma g_t) X_t dt + \sigma X_t dV_t, \\ dY_t &= (\beta + \delta g_t) Y_t dt + \delta Y_t dV_t. \end{cases}$$

In order for $\tilde{Q}$ to be a risk neutral martingale measure both X and Y have to have drifts equal to the short rate r. Thus we need g to satisfy

$$g_t = \frac{r - \alpha}{\sigma}, \quad \text{and} \quad g_t = \frac{r - \beta}{\delta}.$$

This would require the following equation to hold

$$\frac{r - \alpha}{\sigma} = \frac{r - \beta}{\delta}.$$

This however is not possible when $r \neq \frac{\delta\alpha-\sigma\beta}{\delta-\sigma}$. This means that we cannot find a martingale measure for the model and thus it is not free of arbitrage.

Another way to show that there are arbitrage possibilities in the model is of course to construct an arbitrage portfolio.

4. (a) Recall that

$$p(t,T) = e^{-\int_t^T f(t,u)du}.$$

We thus have that

$$p(t,T) = e^{Z(t)},$$

where

$$Z(t) = -\int_t^T f(t,u)du.$$

Using one of the hints we get

$$\begin{aligned} dZ(t) &= f(t,t)dt - \int_t^T df(t,u)du \\ &= r(t)dt - \int_t^T \left[\alpha(t,u)dt + \sigma(t,u)dW_t\right] du \\ &= \left[r(t) - \int_t^T \alpha(t,u)du\right] dt - \left[\int_t^T \sigma(t,u)du\right] dW_t \end{aligned}$$

Finally using Itô's formula on $p(t,T) = \exp\{Z(t)\}$ we get

$$dp(t,T) = \{r(t) + b(t,T)\}p(t,T)dt + a(t,T)p(t,T)dW(t),$$

where

$$\begin{cases} a(t,T) &= -\int_t^T \sigma(t,u)du, \\ b(t,T) &= -\int_t^T \alpha(t,u)du + \frac{1}{2}a^2(t,T). \end{cases}$$

(b) The drift of any price process under the risk neutral martingale measure is equal to the short rate. This means that we have $b(t,T) = 0$. Using the expression for $b(t,T)$ and taking the derivative w.r.t. T we get the HJM drift condition

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)du.$$

(c) We get

$$F(t;T,p(t,S)) = \frac{p(t,S)}{p(t,T)}.$$

(d) Using the forward measure Q^S we have the following pricing formula for a T-claim X

$$\Pi(t;X) = p(t,S)E^S[\frac{X}{p(T,S)}|\mathcal{F}_t],$$

where the super index S indicates that the expectation should be taken under the forward measure Q^S. For the binary asset-or-nothing call we then have

$$\begin{aligned}\Pi(t;Y) &= p(t,S)E^S\left[\left.\frac{p(T,S)I_{\{F(T;T,p(T,S))>K\}}}{p(T,S)}\right|\mathcal{F}_t\right]\\ &= p(t,S)Q^S\left(F(T,T)>K|\mathcal{F}_t\right)\\ &= p(t,S)\left[1-Q^S\left(F(T,T)\le K|\mathcal{F}_t\right)\right],\end{aligned}$$

The Q-dynamics of $F(t,T)=F(t;T,p(T,S))$ are then (use $dp(t,T)=rp(t,T)dt+a(t,T)p(t,T)dV$)

$$\begin{aligned}dF(t,T) &= -\frac{p^S}{(p^T)^2}dp^T+\frac{1}{p^T}dp^s+\frac{1}{2}\frac{2p^S}{(p^T)^3}(dp^T)^2-\frac{1}{(p^T)^2}dp^T dp^S\\ &= \left(a^2(t,T)-a(t,T)a(t,S)\right)F(t,T)dt+\left[a(t,S)-a(t,T)\right]F(t,T)dV(t).\end{aligned}$$

Now use that the Girsanov kernel between Q and Q^S is given by the volatility of the new numeraire $p(t,S)$, i.e. $a(t,S)$, to obtain the Q^S dynamics

$$\begin{aligned}dF(t,T) &= \left(a^2(t,T)-2a(t,T)a(t,S)+a^2(t,S)\right)F(t,T)dt\\ &\quad+\left[a(t,S)-a(t,T)\right]F(t,T)dV^S(t)\\ &= \left[a(t,S)-a(t,T)\right]^2F(t,T)dt+\left[a(t,S)-a(t,T)\right]F(t,T)dV^S(t)\end{aligned}$$

where V^S is a Q^S-Wiener process. Thus

$$\begin{aligned}F(T,T) &= F(t,T)\exp\left\{\int_t^T\frac{1}{2}\left[a(u,S)-a(u,T)\right]^2du+\int_t^T\left[a(u,S)-a(u,T)\right]dV^S(u)\right\}\\ &= F(t,T)e^Z\end{aligned}$$

where $Z\in N\left(\eta^2(t,T)/2,\eta(t,T)\right)$ with η defined by

$$\eta^2(t,T)=\int_t^T\left[a(u,S)-a(u,T)\right]^2du.$$

The price of the binary asset-or-nothing call is therefore

$$\begin{aligned}\Pi(t;X) &= p(t,S)K\left[1-Q^S\left(F(t,T)e^Z\le K\right)\right]\\ &= p(t,S)\left[1-Q^S\left(Z\le\ln\left\{\frac{K}{F(t,T)}\right\}\right)\right]\\ &= p(t,S)\left[1-\Phi\left(\frac{1}{\eta(t,T)}\left\{\ln\left[\frac{K}{F(t,T)}\right]-\frac{1}{2}\eta^2(t,T)\right\}\right)\right]\\ &= p(t,S)\Phi\left(\frac{1}{\eta(t,T)}\left\{\ln\left[\frac{F(t,T)}{K}\right]+\frac{1}{2}\eta^2(t,T)\right\}\right)\\ &= p(t,S)\Phi\left(\frac{1}{\eta(t,T)}\left\{\ln\left[\frac{p(t,S)}{K}\right]-\ln p(t,T)+\frac{1}{2}\eta^2(t,T)\right\}\right),\end{aligned}$$

where we have used one of the hints to obtain the third equality and the fact that $F(t,T)=p(t,S)/p(t,T)$ to obtain the last equality. As usual Φ denotes the cumulative distribution function of the $N(0,1)$ distribution.

5. (a) See the textbook. $F(t;T,X)=E^Q[X|\mathcal{F}_t]$.

(b) Let $H_t = 2F_t - S_t$. We need to check two things in order to show that the rolling spot futures price process is given by $U_t = H_t = 2F_t - S_t$:

i. That $H_T = 2F_T - S_T = S_T$.

ii. That the price process Π and dividend process D of the rolling spot futures contract satisfies the following relation for all $0 \leq t \leq T$

$$\Pi_t = E^Q\left[\exp\left\{-\int_t^T r_s ds\right\}\Pi_T + \int_t^T \exp\left\{-\int_t^s r_u du\right\} dD_s \middle| \mathcal{F}_t\right], \quad (1)$$

given that $U_t = H_t$.

Since we know that the conventional futures price satisfies $F(T;T,S_T) = S_T$, we obviously have that $H_T = 2S_T - S_T = S_T$.

For the other point recall that $\Pi_t \equiv 0$. Given this we see from $G_t = U_t + S_t$ that $D_t = U_t + S_t$. If $U_t = H_t = 2F_t - S_t$ the dividend process becomes $D_t = 2F_t$. Inserting the expressions for Π and D into the right hand side of (1) we obtain

$$Rhs = E^Q\left[\int_t^T \exp\left\{-\int_t^s r_u du\right\} 2dF_s \middle| \mathcal{F}_t\right].$$

We know from (a) that F_t is a martingale, and since we have assumed that the filtration is the natural filtration generated by the Wiener process W, the martingale representation theorem says that F_t can be written as

$$F_t = F_0 + \int_0^t \varphi_s dW_s,$$

for some adapted process φ_t. Thus, the right hand side becomes

$$Rhs = E^Q\left[\int_t^T \exp\left\{-\int_t^s r_u du\right\} 2\varphi_s dW_s \middle| \mathcal{F}_t\right] = 0,$$

where the last equality follows from a property of the stochastic integral. For the left hand side we have

$$Lhs = \Pi_t = 0,$$

and the relation (1) has been proven to hold.