KTH Mathematics

Exam in SF2975 Financial Derivatives.
Tuesday May 20 2008 14.00-19.00.

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Examiner: Camilla Landén, tel 790 8466.

Aids: None.

General instructions: The solutions should be legible, easy to follow and not lacking in explanations. In particular, all notation used should be explained. Problems concerning integrability need not be treated.

N.B. Number the pages and write your name on each sheet. Write only one exercise per sheet and state the number of the exercise on the sheet.

25 points including the bonus guarantees a passing grade.

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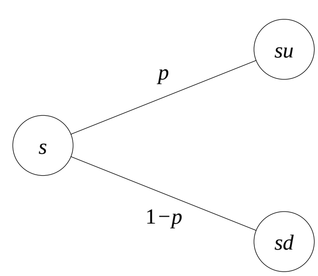
**1.** (a) Compute the price of an *American put option* with strike price $K = 100$ kr and exercise date $T = 2$ years, using a binomial tree with two trading dates $t_1 = 0$ and $t_2 = 1$ (your portfolio at time $t_3 = 2$ is the same as your portfolio at time $t_2 = 1$) and parameters $s_0 = 100$, $u = 1.4$, $d = 0.8$, $r = 10\%$, and $p = 0.75$. ........(3p)

(b) Show that the one period binomial model is complete given that $d < 1+r < u$.




........ (3p)

(c) Consider the standard Black-Scholes model (described in detail in exercise 2).

i. Define what is meant by a **replicating** or **hedging** portfolio for the contingent T-claim $X$. ........(2p)

ii. Consider a portfolio consisting of a long position in a European put option with exercise date $T$ and exercise price $K_1$, and a long position in a European call option with the same exercise date $T$, and exercise price $K_2$, such that $K_1 < K_2$. Is this a replicating portfolio, and if so for which claim? ............................................................... (2p)

**2.** Consider two dates, $T_0$ and $T$, with $T_0 < T$. A *forward-start call option* is a contract in which the holder receives, at time $T_0$ (at no additional cost), a European call option with expiry date $T$ and exercise price equal to $S_{T_0}$. On the other hand, the holder must pay at time 0 an up-front fee, the price of a forward-start call option.

Write down the terminal payoff, i.e. the payoff at time $T$, of a forward-start call option and then determine the arbitrage free price at time $t \in [0, T_0]$ of a forward-start call option. If needed the standard Black-Scholes formula for the price of a European call option can be found on the last page of this exam. ........... (10p)

**3.** Consider an interest rate model described (under a martingale measure $Q$) by

$$dr = \mu(t, r)dt + \sigma(t, r)dV. \tag{1}$$

(a) Define what is meant by an *Affine Term Structure* (ATS), and derive conditions which are sufficient to guarantee the existence of an ATS for the model above. ................................................................. (5p)

(b) Recall that the definition of the continuously compounded zero coupon yield $y(t, T)$ is

$$y(t, T) = -\frac{\ln p(t, T)}{T - t}.$$

Suppose that you want to price a contingent claim which depends on the joint distribution of two yields. It is then of interest to compute the correlation

$$Corr(y(t, T_1), y(t, T_2)),$$

under the risk neutral martingale measure $Q$ for $t < T_1 < T_2$. Do this given that the short rate satisfies

$$dr_t = [b(t) - a(t)r_t]dt + \sigma(t)dV_t,$$

where $V$ is a $Q$-Wiener process. Do you think the correlation structure is satisfactory for pricing purposes? ..................................... (3p)

**Hint:** Use the general structure, and solve equations only if necessary.

(c) When pricing interest rate derivatives it is sometimes useful to use the $T$-forward measure $Q^T$. What is the numeraire asset for the $T$-forward measure, and what does the Girsanov kernel between $Q$ and $Q^T$ look like given that you consider a short rate model possessing an affine term structure? You do not have to derive the transformation between $Q$ and $Q^T$, just state the result, and the expression for the Girsanov kernel may contain $\mu$ and $\sigma$ from equation (1), and functions which can be computed given specific $\mu$ and $\sigma$. .......... (2p)

**4.** Consider a model for two countries. We then have a domestic market (Sweden) and a foreign market (Japan). The domestic and foreign interest rates, $r_d$ and $r_f$, are assumed to be given real numbers. Consequently, the domestic and foreign savings accounts satisfy

$$B_t^d = e^{r_d t}, \qquad B_t^f = e^{r_f t},$$

where $B^d$ and $B^f$ are denominated in units of domestic and foreign currency, respectively. The exchange rate process $X$, which is used to convert foreign payoffs into domestic currency (the "krona/yen"-rate), is modeled by the following stochastic differential equation under the objective measure $P$

$$dX = \alpha_X X dt + \sigma_X X dW.$$

Here $\alpha_X$ and $\sigma_X$ are assumed to be constants and $W$ denotes a $P$-Wiener process.

Suppose that you have entered a financial contract which stipulates that you are to pay a certain amount of money in the foreign currency at a future date. Then you might want to consider buying a *pay later option* on the foreign currency to reduce your exposure to currency risk. The buyer of a pay later option has the obligation to exercise the option when it is in the money and to pay the premium. This means that as soon as the difference between the price of the underlying and the strike price is positive exercise takes place, regardless of how big the difference is, i.e. the amount by which the option is in the money. The payoff at the exercise date $T$ of a pay later option with strike price $K$ written on the foreign currency is thus given by $Y = \phi(X_T)$, where the function $\phi$ is defined by

$$\phi(x) = \begin{cases} x - K - p, & \text{if } x > K, \\ 0, & \text{otherwise.} \end{cases}$$

Here $p$ denotes the premium which the holder of the option must pay upon exercise. The premium $p$ is determined when the contract is initiated (at time $t = 0$) in such a way that the current price of the contract is zero, i.e. $\Pi(0, Y) = 0$.

Your task is to compute the correct premium $p$ (in domestic currency) for the pay later option described above. .............................................. (10p)

**5.** Consider a financial market with two assets. One asset ("the $A$-asset") is a dividend paying asset with price process $A_t$ and cumulative dividend process $D_t$. The other asset ("the $S$-asset") is a non dividend paying asset with price process $S_t$. The asset price processes and the dividend process are Itô processes, driven by a finite number of Wiener processes.

We recall that the normalized gain process

$$\frac{A_t}{B_t} + \int_0^t \frac{1}{B_s} dD_s$$

(where $B$ is the bank account) is a martingale under the risk neutral martingale measure $Q$. The task is now to determine which process (related to $A$) is a martingale under $Q^S$, the martingale measure using $S$ as numeraire process. This can be done in the following steps.

(a) Consider the portfolio where you keep exactly one unit of the $A$-asset and invest all dividends in the $S$-asset. Denote the value of this portfolio at time $t$ by $V_t$ and denote the number of units of asset $S$ at time $t$ by $X_t$.

Write down the value of $V_t$ in terms of $X_t$, $S_t$ and $A_t$. .................. (1p)

(b) Using the fact that $V$ is the value process of a self-financing portfolio, express the differential $dV$ in terms of $X$, $D$, $S$ and $A$ and/or their differentials. (2p)

(c) Now use the expression you gave in (a) to solve for $X$, and then apply the Itô formula to obtain $dX$. Integrate to obtain $X_t$. ......................... (5p)

(d) Finally, use that the value process $V$ is the price process of a non-dividend paying asset, and properties of such processes, to obtain a process related to $A$, which is a martingale under $Q^S$. ...................................... (2p)

*Good luck!*

**Hints:**

You are free to use the following in any of the above exercises.

- The density function of a normally distributed random variable with expectation $m$ and variance $\sigma^2$ is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/(2\sigma^2)}.$$

- Let $N$ denote the cumulative distribution function for the N(0, 1) distribution. Then

$$N(-x) = 1 - N(x).$$

- The standard Black-Scholes formula for the price $\Pi(t)$ of a European call option with strike price $K$ and time of maturity $T$ is $\Pi(t) = F(t, S(t))$, where

$$F(t,s) = s\Phi[d_1(t,s)] - e^{-r(T-t)}K\Phi[d_2(t,s)].$$

Here $\Phi$ is the cumulative distribution function for the $N(0,1)$ distribution and

$$\begin{aligned} d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}, \\ d_2(t,s) &= d_1(t,s) - \sigma\sqrt{T-t}. \end{aligned}$$