KTH Mathematics

Exam in SF2975 Financial Derivatives.
Monday June 2 2008 14.00-19.00.

Examiner: Camilla Landén, tel 790 8466.

Aids: None.

General instructions: The solutions should be legible, easy to follow and not lacking in explanations. In particular, all notation used should be explained. Problems concerning integrability need not be treated.

N.B. Number the pages and write your name on each sheet. Write only one exercise per sheet and state the number of the exercise on the sheet.

25 points including the bonus guarantees a passing grade.

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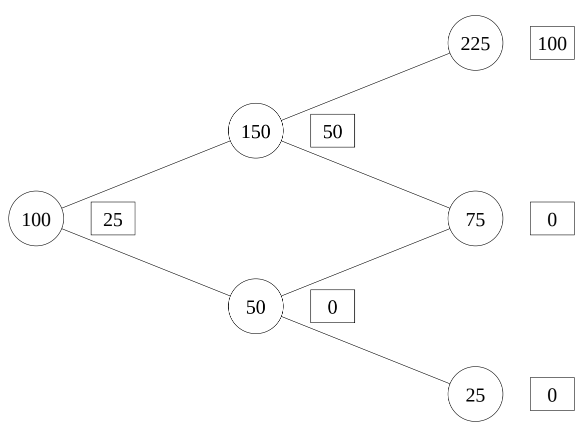
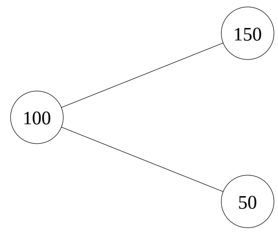
1. (a) In the binomial tree below the price of a European call option with strike price $K = 125$ kr and exercise time $T = 2$ years has been computed using the parameters $s_0 = 100$, $u = 1.5$, $d = 0.5$, $r = 0$, and $p = 0.75$. (The value of the stock is written in the nodes, and the value of the option is written in the adjacent boxes.)



Your task is to find the replicating portfolio for this option and to verify that the portfolio is self-financing. . . . . (3p)

(b) Below is a picture of a one-period (time points $t = 0$ and $t = 1$) binomial model with parameters $s_0 = 100$, $u = 1.5$, $d = 0.5$ and $p = 0.75$.



i. What are the arbitrage bounds for the interest rate r? .... (1p)

ii. Given that the price at time $t = 0$ of a European call option with strike price $K = 108$ kr and exercise time $T = 1$ year has been computed to 22 kr, what is the interest rate r? .... (2p)

(c) Consider a standard Black-Scholes market, i.e., a market consisting of a risk free asset, B, with P-dynamics given by

$$\begin{cases} dB_t &= rB_t dt, \\ B_0 &= 1, \end{cases}$$

and a stock, S, with P-dynamics given by

$$\begin{cases} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ S_0 &= s_0. \end{cases}$$

Here W denotes a P-Wiener process and r, α and σ are assumed to be constants.

i. Check whether the portfolio defined by

$$\mathbf{h}_t = \left(h_t^B, h_t^S\right) = \left(\frac{S_t}{B_t}, \frac{B_t}{S_t}\right),$$

is self-financing or not. .... (2p)

ii. Determine whether the following process X represents a tradable asset or not.

$$X_t = S_t^{-\beta}, \quad \text{where } \beta = 2r/\sigma^2.$$

.... (2p)

2. Consider a standard Black-Scholes market, i.e. a market consisting of a risk free asset, B, with P-dynamics given by

$$\begin{cases} dB_t &= rB_t dt, \\ B_0 &= 1, \end{cases}$$

and a stock, S, with P-dynamics given by

$$\begin{cases} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ S_0 &= s_0. \end{cases}$$

Here W denotes a P-Wiener process and r, α and σ are assumed to be constants.

(a) Determine the arbitrage price of an option which at time T pays either K kr or the value of stock at time T, whichever the holder prefers. The contract function describing the option is thus given by

$$\phi(S_T) = \max\{S_T, K\}.$$

Hint: The easiest way of doing this might be to construct a portfolio consisting of derivatives with known prices, which at time T will pay exactly the same amount as the option above, i.e. to use the same method used to derive put-call-parity. .... (4p)

(b) Now consider an option which has the same contract function as the option in (a), except for that K is replaced by S_{T_0}, where T_0 is a fixed time such that $T_0 < T$. Determine the arbitrage price of this option for $t \in [0, T_0]$. ...... (8p)

3. Consider a two-dimensional Black-Scholes market i.e. a market consisting of a risk-free asset, B, with P-dynamics given by

$$\begin{cases} dB_t &= rB_t dt, \\ B_0 &= 1, \end{cases}$$

and two stocks, X and Y, with P-dynamics given by

$$\begin{cases} dX_t &= \alpha X_t dt + \sigma X_t dW_t, \\ dY_t &= \beta Y_t dt + \delta Y_t dW_t. \end{cases}$$

Here W is a **one**-dimensional P-Wiener processes and r, α, σ, β and δ are assumed to be constants such that

$$r \neq \frac{\delta\alpha - \sigma\beta}{\delta - \sigma}.$$

Assume that the filtration is the natural filtration generated by the Wiener process W. Show that this model is **not** free of arbitrage. .... (4p)

4. A standard HJM model, under the risk neutral martingale measure Q, is of the form

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dV_t, \quad T \geq 0, \tag{1}$$

where V for simplicity is assumed to be a scalar Q-Wiener process.

(a) Given the forward rate dynamics (1) derive the dynamics of the (zero-coupon) bond prices, $p(t,T)$ (you may find some of the formulas on the last page of this exam useful for doing this). .... (4p)

(b) Derive the Heath-Jarrow-Morton drift condition (under Q). .... (1p)

(c) In general the forward price at time t for delivery of X at time T, $F(t;T,X)$ (we use F to denote the forward price, so not to confuse the forward price with the forward rates) is given by

$$F(t;T,X) = \frac{\Pi(t;X)}{p(t,T)} = E^T[X|\mathcal{F}_t]$$

where $\Pi(t;X)$ is the price of X at time t, $p(t,T)$ is the price at time t of a T-bond, and the super index T indicates that the expectation should be taken under the T-forward measure. Give a formula for the forward price at time t for delivery of an S-bond at time T, $F(t;T,p(T,S))$. The formula should be given in terms of prices observable on the market at time t. ....(1p)

(d) A **binary asset-or-nothing call** option written on an S-bond is a T-claim $(T < S)$ with payoff Y given by

$$Y = p(T,S)I_{\{p(T,S)>K\}} = p(T,S)I_{\{F(T,T,p(T,S))>K\}}. \qquad (2)$$

Here I_A is the indicator function of the event A, and K is a prespecified amount of cash.

Compute the arbitrage price of the binary asset-or-nothing call. ....(8p)

Hint: Using the S-forward measure will simplify things. Also, as indicated in Equation (2), the equality $F(T,T,p(T,S)) = p(T,S)$ allows you to do the calculations using the forward price, rather than the spot price, and this will also help simplify the computations.

5. Consider a given market consisting of one risky asset with price process $S(t)$ and cumulative dividend process $D(t)$. The short rate $r(t)$ is assumed to satisfy the following stochastic differential equation

$$dr_t = \mu(t,r_t)dt + \sigma(t,r_t)dW_t.$$

(a) Now consider a fixed contingent T-claim X. Define what is meant by the **futures price process** $F(t;T,X)$. Derive a formula for the futures price process.
....(4p)

(b) Assume that all randomness on the market is generated by the Wiener process W, and that the filtration is the natural filtration generated by this Wiener process. Consider the **rolling spot futures contract**, defined as follows. The contract promises to pay, continually, changes in the spot price S_t of the underlying asset as well as futures resettlement payments. This means that the rolling spot futures price process associated with the contract, denoted by U, contractually satisfies $U_T = S_T$, and that the gain process of the contract is given by $G_t = U_t + S_t$. Just as for the conventional futures contract the market price at time t of the rolling spot futures contract, $\Pi(t)$, is zero for all $t \geq 0$.

Show that, given the existence of an equivalent martingale measure, the rolling spot futures price process is given by $U(t) = 2F(t;T,S_T) - S(t)$, where $F(t;T,S_T)$ is the conventional futures price treated in (a). ....(6p)

Good luck!

Hints:

You are free to use the following in any of the above exercises.

- The density function of a normally distributed random variable with expectation m and variance σ^2 is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/(2\sigma^2)}.$$

- Let N denote the cumulative distribution function for the N(0, 1) distribution. Then

$$N(-x) = 1 - N(x).$$

- The standard Black-Scholes formula for the price $\Pi(t)$ of a European call option with strike price K and time of maturity T is $\Pi(t) = F(t, S(t))$, where

$$F(t, s) = s\Phi[d_1(t, s)] - e^{-r(T-t)} K\Phi[d_2(t, s)].$$

  Here Φ is the cumulative distribution function for the $N(0, 1)$ distribution and

$$\begin{aligned} d_1(t, s) &= \frac{1}{\sigma\sqrt{T-t}} \left\{ \ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t) \right\}, \\ d_2(t, s) &= d_1(t, s) - \sigma\sqrt{T-t}. \end{aligned}$$

- Suppose that there exist processes $X(\cdot, T)$ for every $T \geq 0$ and suppose that Y is a process defined by

$$Y(t) = \int_t^{T_0} X(t, s) ds$$

  Then we have the following version of Itô's formula

$$dY_t = -X(t, t)dt + \int_t^{T_0} dX(t, s)ds.$$