KTH Mathematics

Exam in SF2975 Financial Derivatives.
Monday May 18 2009 08.00-13.00.

Answers and suggestions for solutions.

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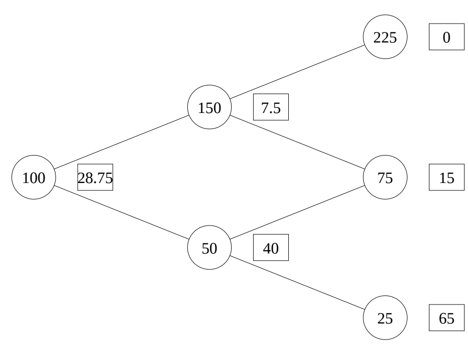
1. (a) For the martingale probabilities we have

$$q = \frac{1+r-d}{u-d} = 0.5$$

Using them we obtain the following binomial tree where the value of the stock is written in the nodes, and the value of the option is written in the adjacent boxes. (The value 40 adjacent to the node with stock price 50 is obtained as $\max\{90-50, 0.5 \cdot 15 + 0.5 \cdot 65\}$.)



The price of the put option is thus 23.75 kr.

(b) The SDE has the solution

$$\begin{aligned} r(t) &= e^{-\beta t} r_0 + \int_0^t e^{-\beta(t-s)} \alpha ds \\ &\quad + \sigma \int_0^t e^{-\beta(t-s)} dW_s. \end{aligned}$$

Thus $r(t)$ is normally distributed with mean $m(t)$

$$\begin{aligned} m(t) &= e^{-\beta t} r_0 + \int_0^t e^{-\beta(t-s)} \alpha ds \\ &= e^{-\beta t} r_0 + \frac{\alpha}{\beta}\left[1 - e^{-\beta t}\right], \end{aligned}$$

and with variance $V(t)$ given by

$$\begin{aligned} V(t) &= E\left[\left\{\sigma\int_0^t e^{-\beta(t-s)}dW_s\right\}^2\right] \\ &= \sigma^2\int_0^t e^{-2\beta(t-s)}ds \\ &= \frac{\sigma^2}{2\beta}\left[1-e^{2\beta t}\right] \end{aligned}$$

(c) i. By definition a portfolio strategy is self-financing if

$$dV(t;\mathbf{h}) = h^B(t)dB(t) + h^S(t)dS(t).$$

An arbitrage strategy is a self-financing strategy $\mathbf{h}$ such that $V(0;\mathbf{h})=0$

$$P(V(T;\mathbf{h})\geq 0) = 1 \quad \text{and} \quad P(V(T;\mathbf{h})>0)>0.$$

A replicating portfolio strategy for a T-claim X is a self-financing portfolio such that

$$V(T;\mathbf{h}) = X, \quad P-a.s.$$

ii. The model is said to be free of arbitrage if there exist no arbitrage strategies, and complete if all claims are reachable, i.e. there exists a replicating portfolio strategy for each claim.

2. (a) If we note that

$$|S_T-K| = 2\max\{S_T-K,0\} - S_T + K,$$

we can write down the price Π of the option as

$$\begin{aligned} \Pi_t &= e^{-r(T-t)}E^Q[2\max\{S_T-K,0\}-S_T+K|\mathcal{F}_t] \\ &= 2c(t,S_t,K,T,r,\sigma) - S_t + e^{-r(T-t)}K. \end{aligned}$$

Here $c(t,s,K,T,r,\sigma)$ denotes the standard Black-Scholes price at time t of a European call option with exercise price K and expiry date T, when the current price of the underlying is s, the interest rate is r, and the volatility of the underlying is σ. Using the hints on the last page of the exam we obtain

$$\Pi_t = 2\left(S_tN[d_1(t,S_t)] - e^{-r(T-t)}KN[d_2(t,S_t)]\right) - S_t + e^{-r(T-t)}K.$$

(b) Developing the square we have

$$X = \phi(S_T) = (S_T^2 - 2KS_T + K^2)^2 I\{S_T > K\}$$

The two last terms you will recognize from the payoff of a standard call option (multiplied by $-2K$ and $-K$, respectively).

The price of the first term is given by

$$\Pi_t = e^{-r(T-t)}E^Q\left[S_T^2 I_{\{S_T>K\}}\middle|\mathcal{F}_t\right].$$

Since $S_T = S_te^Z$ where $Z\in N\left((r-\sigma^2/2)(T-t),\sigma^2(T-t)\right)$ this can be written as

$$\Pi_t = e^{-r(T-t)}\int_{\ln\left\{\frac{K}{S_t}\right\}}^{\infty} S_t^2e^{2z}\varphi(z)dz,$$

where φ denotes the density of a $N\left((r-\sigma^2/2)(T-t), \sigma^2(T-t)\right)$-distribution. Now use that the density function for a $N(m,\sigma^2)$-distributed random variable is $\varphi(z) = e^{-(z-m)^2/(2\sigma^2)}/(\sigma\sqrt{2\pi})$, and then complete the square in the exponent. This yields

$$\Pi_t = e^{-r(T-t)} e^{(2r+\sigma^2)(T-t)} S_t^2 \int_{\ln\left\{\frac{K}{S_t}\right\}}^{\infty} \psi(u)du,$$

where ψ denotes the density of a $N\left((r+3\sigma^2/2)(T-t), \sigma^2(T-t)\right)$-distribution. If we let U denote a $N\left((r+3\sigma^2/2)(T-t), \sigma^2(T-t)\right)$-distributed random variable. Then we have that

$$\begin{aligned}
\Pi_t &= e^{(r+\sigma^2)(T-t)} S_t^2 Q\left(U > \ln\left\{\frac{K}{S_t}\right\}\right) \\
&= e^{(r+\sigma^2)(T-t)} S_t^2 \left[1 - \Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left\{\frac{K}{S_t}\right\} - \left(r + \frac{3}{2}\sigma^2\right)(T-t)\right\}\right)\right] \\
&= e^{(r+\sigma^2)(T-t)} S_t^2 \Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left\{\frac{S_t}{K}\right\} + \left(r + \frac{3}{2}\sigma^2\right)(T-t)\right\}\right),
\end{aligned}$$

where we have used one of the hints to obtain the last equality.

All in all the price is thus given by

$$\Pi^{pc} = S_t^2 e^{(r+\sigma^2)(T-t)}\Phi(d_0(t,S_t)) - 2Ks_t\Phi(d_1(t,S_t)) + e^{-r(T-t)}K^2\Phi(d_2(t,S_t)),$$

where

$$d_i(t,s) = \frac{\ln(s/K) + [r + (3/2 - i)\sigma^2](T-t)}{\sigma\sqrt{T-t}} \quad \text{for } i = 0,1,2.$$

3. (a) Recall that

$$p(t,T) = e^{-\int_t^T f(t,u)du}.$$

We thus have that

$$p(t,T) = e^{Z(t)},$$

where

$$Z(t) = -\int_t^T f(t,u)du.$$

Using one of the hints we get

$$\begin{aligned}
dZ(t) &= f(t,t)dt - \int_t^T df(t,u)du \\
&= r(t)dt - \int_t^T \left[\alpha(t,u)dt + \sigma(t,u)dW_t\right] du \\
&= \left[r(t) - \int_t^T \alpha(t,u)du\right] dt - \left[\int_t^T \sigma(t,u)du\right] dW_t
\end{aligned}$$

Finally using Itô's formula on $p(t,T) = \exp\{Z(t)\}$ we get

$$dp(t,T) = \{r(t) + b(t,T)\}p(t,T)dt + a(t,T)p(t,T)dW(t),$$

where

$$\begin{cases} a(t,T) &= -\int_t^T \sigma(t,u)du, \\ b(t,T) &= -\int_t^T \alpha(t,u)du + \frac{1}{2}a^2(t,T). \end{cases}$$

Thus the bond volatility is given by

$$v(t,T) = -\int_t^T \sigma(t,s)ds.$$

(b) Defining the process Z as

$$Z(t) = \frac{p(t,S)}{p(t,T)},$$

we obtain

$$dZ = Z\left\{v(t,T)^2 - v(t,T)v(t,S)\right\}dt + \{v(t,S) - v(t,T)\}\,dW_t.$$

Standard technique thus gives us

$$E^Q\left[\frac{1}{p(S,T)}\middle|\,\mathcal{F}_t\right] = \frac{p(t,S)}{p(t,T)} \times e^{\int_t^T \left\{v(u,T)^2 - v(u,T)v(u,S)\right\}du},$$

with v as above.

4. (a) On a market with dividend paying assets the normalized gain processes G^Z,

$$G_t^Z = \frac{\Pi_t}{B_t} + \int_0^t \frac{1}{B(s)}dD(s),$$

should be martingales under the riskneutral martingale measure Q. This will result in the following dynamics of S^1 and S^2 under Q

$$\begin{cases} dS_t^1 &= rS_t^1dt + \sigma_1 S_t^1 dV_t^1, \\ dS_t^2 &= (r-\delta)S_t^2dt + \sigma_2\rho S_t^2 dV_t^1 + \sigma_2\sqrt{1-\rho^2}S_t^2dV_t^2. \end{cases}$$

Here V^1 and V^2 are two independent Q-Wiener processes.

(b) From the martingale property of Π/S^1 under the martingale measure using S^1 as numeraire we have that

$$\Pi_t = S_t^1 E^1\left[\frac{\Pi_T}{S_T^1}\middle|\,\mathcal{F}_t\right].$$

Here the superindex 1 indicates that the expectation should be taken under the martingale measure Q^1 using S^1 as numeraire. Using the particular form of the claim we obtain

$$\Pi_t(X) = KS_t^1E^1\left[\max\left\{\frac{S_T^2}{S_T^1} - \frac{1}{K}, 0\right\}\middle|\,\mathcal{F}_t\right].$$

To compute the expectation we need the dynamics of S^2/S^1 under Q^1. We know that the likelihood process L taking us from Q to Q^1 has the from

$$L(t) = \frac{1}{S^1(0)}\frac{S^1(t)}{B(t)}.$$

Applying the Itǒ formula to this we see that L satisfies

$$\begin{cases} dL_t &= \sigma_1 L_t dV_t^1 + 0 \cdot L_t dV_t^2, \\ L_0 &= 1. \end{cases}$$

The dynamics of S^1 and S^2 under Q^1 are thus

$$\begin{cases} dS_t^1 &= (r+\sigma_1^2)S_t^1 dt + \sigma_1 S_t^1 dU_t^1, \\ dS_t^2 &= (r-\delta+\rho\sigma_1\sigma_2)S_t^2 dt + \sigma_2\rho S_t^2 dU_t^1 + \sigma_2\sqrt{1-\rho^2}S_t^2 dU_t^2, \end{cases}$$

where U^1 and U^2 are two independent Q^1-Wiener processes. Now, applying the Itǒ formula to $Z = S^2/S^1$ we obtain

$$\begin{aligned} dZ_t &= -\frac{S_t^2}{(S_t^1)^2}dS_t^1 + \frac{1}{S_t^1}dS_t^2 + \frac{1}{2}2\frac{S_t^2}{(S_t^1)^3}(dS_t^1)^2 - \frac{1}{(S_t^1)^2}dS_t^1 dS_t^2 \\ &= -\delta Z_t dt + (\rho\sigma_2 - \sigma_1)Z_t dU_t^1 + \sigma_2\sqrt{1-\rho^2}Z_t dU_t^2. \end{aligned}$$

This can be written as

$$dZ_t = -\delta Z_t dt + \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}Z_t dU_t,$$

where U is a Q^1-Wiener process. We now have that

$$c(t, Z_t, 1/K, T, -\delta, \sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}) = e^{\delta(T-t)}E^1\left[\max\left\{\frac{S_T^2}{S_T^1} - \frac{1}{K}, 0\right\}\middle| \mathcal{F}_t\right],$$

where $c(t, s, K, T, r, \sigma)$ denotes the standard Black-Scholes price at time t of a European call option with exercise price K and expiry date T, when the current price of the underlying is s, the interest rate is r, and the volatility of the underlying is σ, and thus

$$\Pi_t(X) = KS_t^1 e^{-\delta(T-t)}c(t, Z_t, 1/K, T, -\delta, \sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}).$$

5. (a) The drift of any non-dividend paying, ideally traded price process under the risk neutral martingale measure is equal to the short rate. This means that for $p(t,T)$ we have $b(t,T) = 0$ where b was computed in Exercise 3 (a). Using the expression for $b(t,T)$ and taking the derivative w.r.t. T we get the HJM drift condition

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)du.$$

(b) We have that

$$Z_t = \int_0^t \sigma^2(s,t)ds.$$

where

$$\sigma(t,T) = \eta(r_t)e^{-\int_t^T \kappa(x)dx}.$$

Using the hint we obtain

$$\begin{aligned} Z_t &= \int_0^t \sigma^2(s,t)ds \\ &= \int_0^t \left[\sigma^2(s,s) + \int_s^t \frac{\partial}{\partial u}\sigma^2(s,u)du\right] ds \\ &= \int_0^t \sigma^2(s,s)ds + \int_0^t \left[\int_0^u \frac{\partial}{\partial u}\sigma^2(s,u)ds\right] du \end{aligned}$$

where we have changed the order of integration in the last term. Now, here

$$\sigma(s,s) = \eta^2(r_s) \quad \text{and} \quad \frac{\partial}{\partial u}\sigma^2(s,u) = -2\kappa(u)\sigma^2(s,u),$$

so

$$\begin{aligned} Z_t &= \int_0^t \eta^2(r_s)ds + \int_0^t \left[(-2)\kappa(u)\int_0^u \sigma^2(s,u)ds\right] du \\ &= \int_0^t \eta^2(r_s)ds + \int_0^t (-2)\kappa(u)Z_u du. \end{aligned}$$

Thus,

$$dZ_t = [\eta^2(r_t) - 2\kappa(t)Z_t]dt.$$

For the short rate process we have

$$r(t) = f(t,t) = f(0,t) + \int_0^t \alpha(s,t)ds + \int_0^t \sigma(s,t)dW_s,$$

where

$$\alpha(s,t) = \sigma(s,t)\int_s^t \sigma(s,u)du.$$

Again using the hint we obtain

$$\begin{aligned} r_t &= f(0,t) + \int_0^t \left[\alpha(s,s) + \int_s^t \frac{\partial}{\partial u}\alpha(s,u)du\right] ds \\ &\quad + \int_0^t \left[\sigma(s,s) + \int_s^t \frac{\partial}{\partial u}\sigma(s,u)du\right] dW_s, \end{aligned}$$

Since

$$\begin{aligned} \alpha(s,s) &= 0, & \frac{\partial}{\partial u}\alpha(s,u)du &= -\kappa(u)\alpha(s,u) + \sigma^2(s,u) \\ \sigma(s,s) &= \eta^2(r_s), & \frac{\partial}{\partial u}\sigma(s,u) &= -\kappa(u)\sigma(s,u), \end{aligned}$$

we have

$$\begin{aligned} r_t &= f(0,t) + \int_0^t \left[+\int_s^t \{-\kappa(u)\alpha(s,u) + \sigma^2(s,u)\}du\right] ds \\ &\quad + \int_0^t \left[\eta^2(r_s) + \int_s^t \{-\kappa(u)\sigma(s,u)\}du\right] dW_s, \\ &= f(0,t) + \int_0^t [-\kappa(u)]\left[\int_0^u \alpha(s,u)ds \int_0^u \sigma(s,u)dW_s\right] du + \int_0^t \int_0^u \sigma^2(s,u)dsdu \\ &\quad + \int_0^t \eta^2(r_s)dW_s \end{aligned}$$

where we have changed the order of integration to obtain the last equality. Finally, using that

$$-\kappa(u)\left[\int_0^u \alpha(s,u)ds \int_0^u \sigma(s,u)dW_s\right] = \kappa(u)[f(0,u) - r_u],$$

the definition of Z, and differentiating $f(0,t)$ we find that

$$dr_t = \left\{\frac{\partial}{\partial t}f(0,t) + \kappa(t)[f(0,t) - r_t] + Z_t\right\} dt + \eta^2(r_t)dW_t$$