KTH Mathematics

Exam in SF2975 Financial Derivatives.
Monday August 24 2009 14.00-19.00.

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Aids: None.

General instructions: The solutions should be legible, easy to follow and not lacking in explanations. In particular, all notation used should be explained. Problems concerning integrability need not be treated.

N.B. Number the pages and write your name on each sheet. Write only one exercise per sheet and state the number of the exercise on the sheet.

25 points including the bonus guarantees a passing grade.

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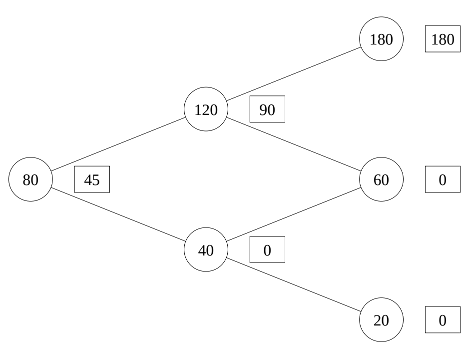
1. (a) In the binomial tree below the price of a binary asset-or-nothing option with expiry in two years and payoff

$$X = \begin{cases} S(2) & \text{if } S(2) > 120, \\ 0 & \text{otherwise,} \end{cases}$$

has been computed using the parameters $s_0 = 80$, $u = 1.5$, $d = 0.5$, $r = 0$, and $p = 0.55$. In the definition of the contract function $S(2)$ denotes the stock price at time $t = 2$. (The value of the stock is written in the nodes, and the value of the option is written in the adjacent boxes.)



Your task is to find the replicating portfolio for this option and to verify that the portfolio is self-financing. .... (3p)

(b) Consider the CIR model for the short rate

$$\begin{aligned} dr(t) &= \{\alpha - \beta r(t)\}\,dt + \sigma\sqrt{r(t)}dW(t), \\ r(0) &= r_0. \end{aligned}$$

Here α, β, and σ are known constants. Compute $E[r(t)]$. .... (3p)

(c) Consider the standard Black-Scholes setting described in detail in Exercise 2. Let K_1 and K_2 be given real numbers such that $0 \le K_1 < K_2$. Now consider the portfolio corresponding to

- buying a call with strike price K_1 and selling a call with strike price K_2
- buying a put with strike price K_2 and selling a put with strike price K_1.

All options are European, written on the stock, and have the same exercise date T. A portfolio strategy like this is known as a *box spread*.

Determine the arbitrage price of the *box spread strategy* described above. (4p)

2. Consider a standard Black-Scholes market, i.e. a market consisting of a risk free asset, B, with P-dynamics given by

$$\begin{cases} dB_t &= rB_t dt, \\ B_0 &= 1, \end{cases}$$

and a stock, S, with P-dynamics given by

$$\begin{cases} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ S_0 &= s_0. \end{cases}$$

Here W denotes a P-Wiener process and r, α and σ are assumed to be constants.

(a) Determine the arbitrage price of a *gap option*, which is a T-claim $X = \phi(S_T)$ with contract function ϕ given by

$$\phi(s) = \begin{cases} 0, & \text{if } s \le K \\ s - A, & \text{otherwise.} \end{cases}$$

The "gap" refers to the difference $A - K$, where both A and K are given real numbers. .... (5p)

(b) Now leave the classical Black-Scholes framework and consider a European call option and an American call option written on the same underlying with the same exercise date. Denote by $c(K)$ the price of the European call option with exercise price K, and by $C(K)$ the price of the corresponding American call option. Let $K_2 > K_1$ be given positive real numbers. Give an arbitrage argument for why the following relation must hold

$$C(K_1) - c(K_1) \ge C(K_2) - c(K_2).$$

The relation states that the incremental value of an American call over its otherwise identical European counterpart is a non-increasing function of the exercise price.

In the classical Black-Scholes framework you would have $c(K) = C(K)$, but for this exercise you may only use the basic properties of the options and should not have to make any assumptions about the underlying. .... (5p)

3. Consider an interest rate model described by

$$dr = \mu(t,r)dt + \sigma(t,r)dV, \tag{1}$$

under a risk-neutral martingale measure Q. Here V denotes a Q-Wiener process.

(a) Define what is meant by an *Affine Term Structure* (ATS), and derive conditions which are sufficient to guarantee the existence of an ATS for the model above. .... (5p)

(b) For Gaussian short rate models with an affine term structure it is possible to obtain closed-form solutions for the prices of European options written on a zero coupon bond. The purpose of this exercise is to see that this means that also the prices of caplets (and therefore of caps) can be given on closed-form.

Let $p(t,T)$ denote the price at time t of a zero coupon bond maturing at time T. Fix two points in time S and T, such that $S < T$ and recall that the payoff X of a *caplet* at time T is

$$\begin{aligned} X &= (T-S)\cdot \max\{L(S,T)-R,0\} \\ &= (T-S)\cdot \max\left\{\frac{1-p(S,T)}{(T-S)p(S,T)} - R, 0\right\}. \end{aligned}$$

where $L(S,T)$ denotes the LIBOR rate contracted at time S for the period $[S,T]$ and the cap rate R is a constant.

Show that the caplet can be replicated using a portfolio of European put options written on zero coupon bonds. .... (5p)

4. Consider a model for two countries. We then have a domestic market (Sweden) and a foreign market (Japan). The domestic and foreign interest rates, r_d and r_f, are assumed to be given real numbers. Consequently, the domestic and foreign savings accounts satisfy

$$B_t^d = e^{r_d t}, \qquad B_t^f = e^{r_f t},$$

where B^d and B^f are denominated in units of domestic and foreign currency, respectively.

On the market there is also an exchange rate process X, which is used to convert foreign payoffs into domestic currency (the "krona/yen"-rate), and a foreign equity with (foreign) price S^f. A domestic/foreign price is the price in units of the domestic/foreign currency. The dynamics of these two processes under the objective measure P are given by

$$\begin{aligned} dX &= X\alpha_X dt + X\sigma_{X1}dW_1 + X\sigma_{X2}dW_2 \\ &= X\alpha_X dt + X\sigma_X d\overline{W}, \end{aligned}$$

$$\begin{aligned} dS^f &= S^f\alpha_f dt + S^f\sigma_{f1}dW_1 + S^f\sigma_{f2}dW_2 \\ &= S^f\alpha_f dt + S^f\sigma_f d\overline{W}, \end{aligned}$$

where

$$\overline{W} = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix},$$

is a two-dimensional P-Wiener process. Furthermore, α_X, and α_f are assumed to be constants, and σ_X and σ_f are linearly independent, constant, row vectors given by

$$\sigma_X = (\sigma_{X1}, \sigma_{X2}), \quad \text{and} \quad \sigma_f = (\sigma_{f1}, \sigma_{f2}).$$

A *domestic martingale measure*, Q^d, is a measure which is equivalent to the objective measure P, and which makes all a priori given price processes, expressed in units of *domestic* currency, and discounted using the *domestic* risk-free rate, martingales.

We assume that if you buy the foreign currency this is immediately invested in a foreign bank account. All markets are assumed to be frictionless.

(a) Determine the Q^d-dynamics of S^f. ....(5p)

Hint: Since there are two Wiener processes the likelihood process will be of the form

$$\begin{aligned} dL(t) &= L(t)\varphi_1(t)dW_1(t) + L(t)\varphi_2(t)dW_2(t) \\ &= L(t)\varphi^*(t)d\overline{W}, \end{aligned}$$

where $\varphi(t) = (\varphi_1(t), \varphi_2(t))^*$, and $*$ denotes transpose. To actually compute φ_1 and φ_2 is rather messy, but fortunately solving for $\sigma_X\varphi$ and $\sigma_f\varphi$ is easier.

(b) Assume that an investor believes that the foreign stock will appreciate and wants to receive any positive returns from the foreign market. However he wants to be sure that those returns are still meaningful when translated back into his own currency. He might then be interested in a *foreign equity call struck in domestic currency.* The payoff Y at expiry T of such an option is given by

$$Y = \max\{X_T S_T^f - K^d, 0\},$$

where the strike price K^d is expressed in **domestic** currency.

Determine the arbitrage price of the option described above expressed in units of domestic currency. ....(5p)

5. Consider the following short rate model under an equivalent martingale measure Q

$$dr_t = qr^{3/2}dV_t.$$

Here q is a constant, $0 < q < 1$, and V is a one-dimensional standard Q-Wiener process. A *consol* is a claim to a constant dividend rate process, say 1, perpetually. This means that the holder of a consol receives 1 dollar per time unit for all time. It can be shown that the price of a consol at time t, $\Pi(t)$, is of the form

$$\Pi(t) = g(r_t) = \frac{A}{r_t}.$$

Your task is to compute the constant A. .... (10p)

Hint: One way of doing this is to recall the fact that the Z-gain process G^Z (the normalized gain process) is a Q-martingale and to use this fact to derive an ordinary differential equation for g.

Hints:
You are free to use the following in any of the above exercises.

- The density function of a normally distributed random variable with expectation m and variance σ^2 is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-m)^2/(2\sigma^2)}.$$

- Let Φ denote the cumulative distribution function for the $N(0,1)$ distribution. Then

$$\Phi(-x) = 1 - \Phi(x).$$

- The standard Black-Scholes formula for the price $\Pi(t)$ of a European call option with strike price K and time of maturity T is $\Pi(t) = F(t, S(t))$, where

$$F(t,s) = s\Phi[d_1(t,s)] - e^{-r(T-t)}K\Phi[d_2(t,s)].$$

  Here Φ is the cumulative distribution function for the $N(0,1)$ distribution and

$$\begin{aligned} d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}, \\ d_2(t,s) &= d_1(t,s) - \sigma\sqrt{T-t}. \end{aligned}$$

Good luck!