KTH Matematik

SOLUTION TO EXAMINATION IN SF2975 FINANCIAL DERIVATIVES 2010-05-24.

Problem 1

(a) See Björk's book, Chapter 2.
(b) Since $\max(S_T - K, 0) = S_T - K + \max(K - S_T, 0)$ it follows by linearity of the price Π that

$$\Pi(\max(S_T - K, 0)) = \Pi(S_T) - \Pi(K) + \Pi(K - S_T, 0).$$

In the Black-Scholes model this becomes

$$c(t, T, S_0, K, r, \sigma) = S_0 - e^{-r(T-t)}K + p(t, T, S_0, K, r, \sigma),$$

where c is the price of a call and p the price of a put.
(c) The martingale measure Q given by Girsanov's theorem with kernel φ. To determine φ, consider a self-financing portfolio $h = (h^0, 1)$ with one risky asset. Then

$$dV^h = \{\text{Itô}\} = h^0 dB + B dh^0 + dS,$$
$$dV^h = \{\text{self-fin}\} = h^0 dB + dS + dD,$$

which leads to $dh^0 = (1/B)dD$. Since V^h is a price process of a self-financing portfolio it must have drift r under Q. We have

$$dV^h = (rh^0 B + (\mu + \delta + \sigma\varphi)S)dt + \ldots dW^Q,$$

which leads to $\varphi = (r - \mu - \delta)/\sigma$. Then, it follows that

$$dS = S(\mu + \sigma\varphi)dt + S\sigma dW^Q = S(r - \delta)dt + S\sigma dW^Q.$$

Problem 2

(a) Let Q be given by $dQ/dP = L_T$ on $\mathcal{F}_T$ where $dL_t = L_t \varphi_t dW_t$, $L_0 = 1$. By Girsanov's theorem it follows that $dW_t = dW^Q + \varphi_t dt$ where W^Q is a Brownian motion under Q. If $L_t > 0$ for $0 \leq t \leq T$ then Q is equivalent to P.
With $\varphi_t = (r - \mu)/\sigma$, it follows that

$$\begin{aligned}
d\frac{S}{B} &= \frac{1}{B}dS - \frac{S}{B^2}dB \\
&= \frac{S}{B}(\mu - r)dt + \frac{S}{B}\sigma dW_t \\
&= \frac{S}{B}(\mu - r + \sigma\varphi)dt + \frac{S}{B}\sigma dW_t^Q \\
&= \frac{S}{B}\sigma dW_t^Q.
\end{aligned}$$

Then S/B is a Q-martingale and we have proved that there exists an equivalent martingale measure.
(b) Let V^h be the value process of a self-financing portfolio with $P(V_T^h \geq 0) = 1$ and $P(V_T^h > 0) > 0$. Then, since Q is equivalent to P, $Q(V_T^h \geq 0) = 1$ and $Q(V_T^h > 0) > 0$ as well. In particular, $E^Q[V_T^h] > 0$. But then, since V^h is self-financing

$$d\frac{V_T^h}{B} = h^1 d\frac{S}{B} = h^1 \frac{S}{B}\sigma dW_t^Q,$$

is an Itô integral and hence a Q-(local)-martingale. In particular

$$V_0^h = \frac{V_0^h}{B_0} = \frac{E^Q[V_T^h]}{B_T} > 0.$$

Hence, h cannot be an arbitrage.

Problem 3

(a) Consider a forward rate model of the form

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW^Q(t),$$

under a measure Q. For Q to be a martingale measure (with bank account as numeraire) it is neccessary that the parameters must satisify the relation

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma^*(t,s)ds.$$

This is the HJM condition. The condition follows since all discounted price processes must be martingales under Q. In particular all zero-coupon bond prices $p(t,T) = \exp\{-\int_t^T f(t,s)ds\}$ must have drift $r(t) = f(t,t)$. To derive the condition the dynamics of $p(t,T)$ is derived from the dynamics of f. Putting the drift equal to $r(t)$ leads to a relation (in integrated form) for the parameters α and σ. The HJM condition is obtained by differentiating that relation w.r.t. T.
(b) On integrated form

$$\begin{aligned}
f(t,T) &= f(0,t) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW^Q(s) \\
&= f(0,0) + \int_0^t f_T(0,u)du + \int_0^t \alpha(s,s) + \int_s^T \alpha_T(s,u)du \\
&\quad + \int_0^t \sigma(s,s) + \int_s^T \sigma_T(s,u)dudW^Q(s).
\end{aligned}$$

With $T = t$ we find

$$
\begin{aligned}
r(t) &= f(t,t) \\
&= r(0) + \int_0^t f_T(0,u)du + \int_0^t \alpha(s,s)ds + \int_0^t \int_s^t \alpha_T(s,u)duds \\
&\quad + \int_0^t \sigma(s,s)dW^Q(s) + \int_0^t \int_s^t \sigma_T(s,u)dudW^Q(s) \\
&= r(0) + \int_0^t f_T(0,u)du + \int_0^t \alpha(u,u)du + \int_0^t \int_0^u \alpha_T(s,u)dsdu \\
&\quad + \int_0^t \sigma(u,u)dW^Q(u) + \int_0^t \int_0^s \sigma_T(s,u)dW^Q(s)du \\
&= r(0) + \int_0^t \Big[f_T(0,u) + \int_0^u \alpha_T(s,u)ds + \int_0^s \sigma_T(s,u)dW^Q(s)\Big]du \\
&\quad + \int_0^t \alpha(u,u)du + \int_0^t \sigma(u,u)dW^Q(u) \\
&= r(0) + \int_0^t \alpha(u,u) + f_T(u,u)du + \int_0^t \sigma(u,u)dW^Q(u).
\end{aligned}
$$

Thus, $a(t) = \alpha(t,t) + f_T(t,t)$ and $b(t) = \sigma(t,t)$. The drift condition, with $T = t$ implies $\alpha(t,t) = 0$ but does not imply a particular relation between $a(t)$ and $b(t)$.

Another way to see it is to let $a(t)$ and $b(t)$ be arbitrary and $F(t, r(t), T) = p(t,T)$ is the price of a zero-coupon T-bond at time t. Then F must satisfy the term-structure-equation, which eventually implies

$$dp(t,T) = r(t)p(t,T)dt + b(t)\partial_r F(t, r(t), T)dW^Q(t).$$

Going from dynamics of $p(t,T)$ to $f(t,T)$ it follows that

$$
\begin{aligned}
\alpha(t,T) &= b(t)\partial_r F(t, r(t), T) \cdot b(t)\partial_T\partial_r F(t, r(t), T), \\
\sigma(t,T) &= -b(t)\partial_T\partial_r F(t, r(t), T),
\end{aligned}
$$

Since $\int_t^T b(t)\partial_T\partial_r F(t, r(t), s)ds = b(t)\partial_r F(t, r(t), T)$ the HJM condition is automatically satisfied without any particular relation between $a(t)$ and $b(t)$.

Problem 4

(a) Note that the payoff in domestic currency is $X(T)B_f(T) - e^{\bar{r}T}$. The amount $X(T)B_f(T)$ can be replicated perfectly by putting 1 unit of foreign currency into the foreign bank account at time $t = 0$. Therefore the value of this part of the contract is X_0 in domestic currency. The fixed amount $e^{\bar{r}T}$ at time T is equivalent to $p_d(0,T)e^{\bar{r}T}$ at time $t = 0$. Therefore $\bar{r}$ is the solution to

$$X_0 - p_d(0,T)e^{\bar{r}T} = 0,$$

which implies

$$\bar{r} = \frac{1}{T}\log\Big(\frac{X_0}{p_d(0,T)}\Big).$$

(b) Since the contract is replicated by the portfolio with one unit in the foreign bank accunt and a short position of a domestic T-bond with nominal $e^{\bar{r}T}$ it follows that $V(t) = \tilde{B}_f(t) - p_d(t,T)e^{\bar{r}T}$, where $\tilde{B}_f(t) = X(t)B_f(t)$. Since, under Q^d, the drift of $\tilde{B}_f(t)$ must be equal to $r_d(t)$ we have

$$d\tilde{B}_f(t) = X(t)dB_f(t) + B_f(t)dX(t) = \tilde{B}_f(t)r_d(t)dt + \tilde{B}_f(t)\sigma_X(t)dW^{Q^d}(t).$$

Then,

$$\begin{aligned} dV(t) &= d(\tilde{B}_f(t)) + e^{\bar{r}T}dp_d(t,T) \\ &= V(t)r_d(t)dt + V(t)\Big(\frac{\tilde{B}_f(t)\sigma_X(t) + p_d(t,T)v_d(t,T)}{V(t)}\Big)dW^{Q^d}(t), \end{aligned}$$

and

$$\begin{aligned} a(t) &= r_d(t), \\ b(t) &= \frac{\tilde{B}_f(t)\sigma_X(t) + p_d(t,T)v_d(t,T)}{V(t)}. \end{aligned}$$

Problem 5

First note that

$$\max(K\tilde{S}_T^f - S_T^d, 0) = K\tilde{S}_T^t I\{Z_T \geq 1/K\} - S_T^d I\{Z_T \geq 1/K\}.$$

Using the general formula $\Pi(X) = N(t)E^{Q^N}[Z/N_T \mid \mathcal{F}_t]$ for pricing a T-claim X, where N is the numeraire it follows that

$$\begin{aligned} \Pi(\max(K\tilde{S}_T^f - S_T^d, 0)) &= K\tilde{S}_t^f E^{\tilde{Q}^f}[I\{Z_T \geq 1/K\}] + S_t^d E^{Q^d}[I\{Z_T \geq 1/K\}] \\ &= K\tilde{S}_t^f \tilde{Q}^f(Z_T \geq 1/K) + S_t^d Q^d(Z_T \geq 1/K). \end{aligned}$$

(b) The dynamics of Z_t is, by Itô's formula,

$$dZ_t = Z_t\{-(\sigma_f + \sigma_X)\sigma_d^* + \sigma_d\sigma_d^*\}dt + Z_t\{\sigma_f + \sigma_X - \sigma_d\}dW_t^Q.$$

The Girsanov transformation from Q to $\tilde{Q}^f$ has kernel $\varphi = \sigma_f + \sigma_X$ which implies $dW^Q = dW^{\tilde{Q}^f} + (\sigma_f + \sigma_X)^* dt$. Then

$$dZ_t = Z_t(\sigma_f + \sigma_X - \sigma_d)((\sigma_f + \sigma_X - \sigma_d)^* dt + Z_t(\sigma_f + \sigma_X - \sigma_d)dW_t^{\tilde{Q}^f}.$$

Put $\sigma^2 = \|\sigma_f + \sigma_X - \sigma_d\|^2$. Then, Z_T under $\tilde{Q}^f$ can be represented as

$$Z_T = Z_t \exp\{(\sigma^2 - \sigma^2/2)(T-t) + \sigma\sqrt{T-t}N(0,1)\},$$

where $N(0,1)$ is a standard normal distribution. In particular

$$\begin{aligned} \tilde{Q}^f(Z_T \geq 1/K) &= \text{Prob}(\sigma^2(T-t)/2 + \sigma\sqrt{T-t}N(0,1) \geq -\log K - \log Z_t) \\ &= \text{Prob}(N(0,1) \leq d_1), \end{aligned}$$

with $d_1 = \frac{\log K + \log Z_t + \sigma^2(T-t)/2}{\sqrt{\sigma^2(T-t)}}$. Similarly, the Girsanov transformation from Q to Q^d has kernel σ_d and following the outlined procedure

$$Q^d(Z_T \geq 1/K) = \text{Prob}(N(0,1) \leq d_2),$$

with $d_2 = \frac{\log K + \log Z_t + \sigma_d\sigma_d^*(T-t) - \sigma^2(T-t)/2}{\sqrt{\sigma^2(T-t)}}$.