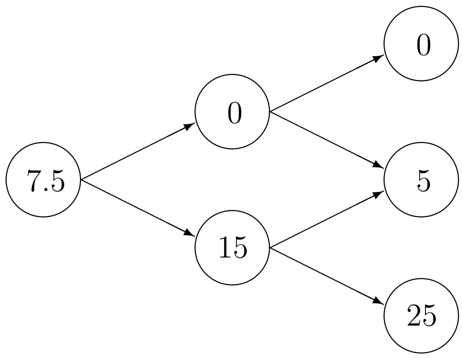
KTH Matematik

SOLUTION TO EXAMINATION IN SF2975 FINANCIAL DERIVATIVES 2010-08-26.

Problem 1

(a) By constructing a replicating portfolio the value of the option in each node of the tree is given by



Hence, the price of the option is 7.5.
(b) By drawing the payoff function of both portfolios it is seen that the payoff of portfolio A dominates B. Hence, by no-abritrage the portfolio A must be more expensive.
(c) The market is free of arbitrage if and only if there exists an equivalent martingale measure Q; such that for each asset $i = 1, \dots, d$, $S^{(i)}(t)/B(t)$ is a Q-martingale.

Problem 2

The price at time $t < T_0$ is given by

$$\begin{aligned}
\Pi(t;X) &= e^{-r(T_0-t)}E^Q[\max\{c(S_{T_0}, T-T_0, S_{T_0}) - K_0, 0\} \mid \mathcal{F}_t] \\
&= e^{-r(T_0-t)}E^Q[\max\{e^{-r(T-T_0)}E^Q[\max\{S_T - S_{T_0}, 0\} \mid F_{T_0}] - K_0, 0\} \mid \mathcal{F}_t] \\
&= e^{-r(T_0-t)}E^Q[\max\{e^{-r(T-T_0)}S_{T_0}E^Q[\max\{\frac{S_T}{S_{T_0}} - 1, 0\} \mid F_{T_0}] - K_0, 0\} \mid \mathcal{F}_t] \\
&= e^{-r(T_0-t)}E^Q[\max\{S_{T_0}c(1, T-T_0, 1) - K_0, 0\} \mid \mathcal{F}_t] \\
&= c(1, T-T_0, 1)e^{-r(T_0-t)}E^Q[\max\{S_{T_0} - \frac{K_0}{c(1, T-T_0, 1)}, 0\} \mid \mathcal{F}_t] \\
&= c(1, T-T_0, 1)c(S_t, T_0 - t, K_0/c(1, T-T_0, 1)).
\end{aligned}$$

Problem 3

Consider a market with stochastic short rate $r(t)$, a bank account $B(t) = \exp\{\int_0^t r(s)ds\}$, a T-bond $p(t,T)$ and a stock $S(t)$. Let $X = I\{S(T) \geq K\}$. Then, with the bank account B as numeraire

$$\Pi(0;X) = E^{Q^B}[\exp\{-\int_0^T r(t)dt\} I\{S(T) \geq K\}],$$

which looks complicated unless r and S are independent. However, with $p(t,T)$ as numeraire we get, since $p(T,T) = 1$

$$\Pi(0;X) = p(0,T)E^{Q^T}[I\{S(T) \geq K\}] = p(0,T)Q^T(S(T) \geq K),$$

which is much easier.

Problem 4

It is assumed that $F^T(t,r)$ is sufficiently smooth. Then Ito's formula implies

$$\begin{aligned} dF^T(t,r(t)) &= \partial_t F^T(t,r(t))dt + \partial_r F^T(t,r(t))dr(t) + \frac{1}{2}\partial_{rr}F^T(t,r(t))dr(t)dr(t) \\ &= \Big(\partial_t F^T(t,r(t)) + \mu(t,r(t))\partial_r F^T(t,r(t)) + \frac{1}{2}\sigma^2(t,r(t))\partial_{rr}F^T(t,r(t))\Big)dt \\ &\quad + \sigma(t,r(t))\partial_r F^T(t,r(t))dW(t) \\ &= m^T(t,r(t))F^T(t,r(t))dt + v^T(t,r(t))F^T(t,r(t))dW(t), \end{aligned}$$

with

$$\begin{aligned} m^T(t,r) &= \frac{\partial_t F^T(t,r) + \mu(t,r)\partial_r F^T(t,r) + \frac{1}{2}\sigma^2(t,r)\partial_{rr}F^T(t,r)}{F^T(t,r)}, \\ v^T(t,r(t)) &= \frac{\sigma(t,r)\partial_r F^T(t,r)}{F^T(t,r)}. \end{aligned}$$

Existence of an equivalent martingale measure Q implies (with the converse of Girsanov) that there is a stochastic process $\varphi(t)$ (that does not depend on T) such that $dQ/dP = L$ with $dL(t) = \varphi(t)L(t)dW(t)$ and $F^T(t,r(t))$ has drift $r(t)$ under Q. That is,

$$\varphi(t) = \frac{r(t) - m^T(t,r(t))}{v^T(t,r(t))},$$

for each T. This can be rerwitten as

$$\begin{aligned} 0 &= m^T(t,r) + v^T(t,r)\varphi(t) - r \\ &= \frac{\partial_t F^T(t,r) + \mu(t,r)\partial_r F^T(t,r) + \frac{1}{2}\sigma^2(t,r)\partial_{rr}F^T(t,r)}{F^T(t,r)} + \frac{\sigma(t,r)\partial_r F^T(t,r)}{F^T(t,r)}\varphi(t) - r. \end{aligned}$$

which implies

$$\partial_t F^T(t,r) + (\mu(t,r) + \sigma(t,r)\varphi(t))\partial_r F^T(t,r) + \frac{1}{2}\sigma^2(t,r)\partial_{rr}F^T(t,r) = rF^T(t,r),$$

and the terminal condition $F^T(T,r) = 1$. This is the Term Structure Equation.

Problem 5

(a) The dynamics under Q are given by

$$dS^{(1)}(t) = (r - \delta^{(1)})S^{(1)}dt + S^{(1)}(t)\sigma_1 dW^Q(t),\ S_0^{(1)} = s_1,$$
$$dS^{(2)}(t) = (r - \delta^{(2)})S^{(2)}dt + S^{(2)}(t)\sigma_2 dW^Q(t),\ S_0^{(2)} = s_2.$$

To see this, let Q be given by a Girsanov transformation with kernel φ and note that a self-financing portfolio (h^0, h^1, h^2) has value process

$$\begin{aligned}
dV^h &= \Big(h^{(0)}Br + h^{(1)}S^{(1)}(\mu^{(1)} + \delta^{(1)}) + h^{(2)}S^{(2)}(\mu^{(2)} + \delta^{(2)})\Big)dt \\
&\quad + \Big(h^{(1)}S^{(1)}\sigma_1 + h^{(2)}S^{(2)}\sigma_2\Big)dW \\
&= \Big(h^{(0)}Br + h^{(1)}S^{(1)}(\mu^{(1)} + \delta^{(1)} + \sigma_1\varphi^T) + h^{(2)}S^{(2)}(\mu^{(2)} + \delta^{(2)}) + \sigma_2\varphi^T\Big)dt \\
&\quad + \Big(h^{(1)}S^{(1)}\sigma_1 + h^{(2)}S^{(2)}\sigma_2\Big)dW^Q.
\end{aligned}$$

By no-abritrage V^h must have drift r under Q which implies φ must solve

$$\mu^{(1)} + \delta^{(1)} + \sigma_1\varphi^T = r$$
$$\mu^{(2)} + \delta^{(2)} + \sigma_2\varphi^T = r.$$

Since σ is invertible a unique solution exists. Then, the $S^{(1)}$ dynamics are equal to

$$\begin{aligned}
dS^{(1)}(t) &= (\mu^{(1)} + \sigma_1\varphi^T)S^{(1)}dt + S^{(1)}(t)\sigma_1 dW^Q(t) \\
&= (r - \delta^{(1)})S^{(1)}dt + S^{(1)}(t)\sigma_1 dW^Q(t),
\end{aligned}$$

and similarly for $S^{(2)}$.

(b) We need to find α such that

$$0 = e^{-rT}E^Q[D_T^{(1)} - \alpha D_T^{(2)}].$$

First compute

$$\begin{aligned}
E^Q[D_T^{(1)}] &= E^Q\Big[\int_0^T \delta^{(1)}S^{(1)}(t)dt\Big] \\
&= \delta^{(1)}\int_0^T E^Q[S_t^{(1)}]dt \\
&= \delta^{(1)}s_1\int_0^T E^Q[\exp\{(r - \delta^{(1)} - \|\sigma_1\|^2/2)t + \|\sigma_1\|W^Q(t)\}]dt \\
&= \delta^{(1)}s_1\int_0^T \exp\{(r - \delta^{(1)})t\}dt \\
&= \frac{\delta^{(1)}s_1}{r - \delta^{(1)}}\Big(\exp\{(r - \delta^{(1)})T\} - 1\Big).
\end{aligned}$$

Similarly

$$E^Q[D_T^{(2)}] = \frac{\delta^{(2)} s_2}{r - \delta^{(2)}}\Big(\exp\{(r - \delta^{(2)})T\} - 1\Big),$$

which gives

$$\alpha = \frac{E^Q[D_T^{(1)}]}{E^Q[D_T^{(2)}]} = \frac{\delta^{(1)} s_1 (r - \delta^{(2)})(\exp\{(r - \delta^{(1)})T\} - 1)}{\delta^{(2)} s_2 (r - \delta^{(1)})(\exp\{(r - \delta^{(2)})T\} - 1)}.$$