KTH Matematik

EXAMINATION IN SF2975 FINANCIAL DERIVATIVES, 2010-05-24, 08:00–13:00.

Examiner: Henrik Hult, tel. 790 6911, e-mail: hult@kth.se

Allowed technical aids: none.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

Bonus points: $10-12$ bonus point give full credit for Problem 1. $7-9$ bonus point give full credit for Problem 1a and 1b. $4-6$ bonus points give full credit for Problem 1a.

GOOD LUCK!

- -

Problem 1

(a) Define the one-period binomial model consisting of a bank account and a risky asset. Introduce all notation needed. Show that the one-period binomial model is free of arbitrage and complete. (4 p)

(b) State and prove the put-call-parity. (3 p)

(c) Consider a Black-Scholes model with a bank account B and a dividend paying stock S with dividend process D such that

$$\begin{aligned} dB_t &= rB_t dt,\ B_0 = 1, \\ dS_t &= \mu S_t dt + \sigma S_t dW_t,\ S_0 = s, \\ dD_t &= \delta S_t dt. \end{aligned}$$

Suppose the market is free of arbitrage. Derive the dynamics of S under the equivalent martingale measure Q (with B as numeraire). (3 p)

Problem 2

Consider a standard Black-Scholes model

$$\begin{aligned} dB_t &= rB_t dt,\ B_0 = 1, \\ dS_t &= \mu S_t dt + \sigma S_t dW_t,\ S_0 = s. \end{aligned}$$

where W is a one-dimensional Brownian motion, μ and σ are constants, and the filtration is the one generated by the Brownian motion. Show that the Black-Scholes model is **free of arbitrage** by showing

(a) there exists an equivalent martingale measure Q, (5 p)

(b) existence of an equivalent martingale measure Q implies no arbitrage. (5 p)

Problem 3

Consider a standard HJM model under the risk neutral martingale measure Q of the form

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW^Q(t),$$

where W^Q is a one-dimensional Brownian motion.

(a) Explain the HJM **drift-condition**. What does it say? Why is it needed? How is it derived (only sketch is needed)? (5 p)

(b) Does the HJM drift-condition imply restrictions on the parameters of the short rate under Q?
To answer this question: derive the short rate dynamics from the forward rate dynamics. That is, determine the processes $a(t)$ and $b(t)$ such that

$$dr(t) = a(t)dt + b(t)dW^Q(t).$$

Does the HJM drift-condition imply a relation between $a(t)$ and $b(t)$? (5 p)

Problem 4

Consider an international bond market where $r_d(t)$ is the domestic short rate, $B_d(t)$ is the domestic bank account, and $p_d(t,T)$ refers to the domestic zero-coupon bond prices. There is also a foreign short rate $r_f(t)$, a foreign bank account $B_f(t)$, and foreign zero-coupon bond prices $p_f(t,T)$, all noted in the foreign currency. The exchange rate is denoted X_t.
Suppose the domestic zero-coupon bond prices have dynamics given by

$$dp_d(t,T) = p_d(t,T)r_d(t)dt + p_d(t,T)v_d(t,T)dW^{Q^d}(t),$$

where W^{Q^d} is a k-dimensional Brownian motion under a martingale measure Q^d with the domestic bank account $B_d(t)$ as numeraire.
Suppose the foreign zero-coupon bond prices are given by

$$dp_f(t,T) = p_f(t,T)m(t,T)dt + p_f(t,T)v_f(t,T)dW^{Q^d}(t),$$

and the exchange rate

$$dX(t) = X(t)\mu(t)dt + X(t)\sigma_X(t)dW^{Q^d}(t).$$

Suppose the market is free of arbitrage. The no-arbitrage assumption implies that $\mu(t)$ and $m(t,T)$ cannot be arbitrary (but you don't have to establish the 'drift condition').

Consider a swap-type contract, written at $t = 0$, where the holder receives at time T the amount $\exp\{\int_0^T r_f(s)ds\}$ in the foreign currency and has to pay the amount $\exp\{\bar{r}T\}$ in domestic currency.

(a) Determine $\bar{r}$ such that the contract at time $t = 0$ has value 0. (5 p)

(b) Suppose you enter the contract at time $t = 0$. At a future time $t \in (0, T)$ the contract has value $V(t)$ which may be positive or negative. Determine the dynamics of $V(t)$ under Q^d. That is, determine the processes $a(t)$ and $b(t)$ such that

$$dV(t) = V(t)a(t)dt + V(t)b(t)dW^{Q^d}(t).$$

(5 p)

Problem 5

Consider a market with a domestic bank account B^d, a domestic stock S^d, a foreign bank account B^f, an exchange rate X, and a foreign stock S^f. Suppose the market is free of arbitrage and the Q-dynamics are given by (Q refers to the martingale measure with B^d as numeraire)

$$\begin{aligned}
dB_t^d &= r_d B_t^d dt, \ B_0^d = 1, \\
d\tilde{B}_t^f &= r_d \tilde{B}_t^f dt + \tilde{B}_t^f \sigma_X dW_t^Q, \ \tilde{B}_0^f = 1, \\
dX_t &= X_t(r_d - r_f)dt + X_t \sigma_X dW_t^Q, \ X_0 = x, \\
d\tilde{S}_t^f &= \tilde{S}_t^f r_d dt + \tilde{S}_t^f (\sigma_f + \sigma_X) dW_t^Q, \ \tilde{S}_0^f = s^f x, \\
dS_t^d &= S_t^d r_d dt + S_t^d \sigma_d dW_t^Q, \ S_0^d = s^d,
\end{aligned}$$

where W_t^Q is a k-dimensional Brownian motion, σ_d, σ_f, and σ_X are deterministic k-dimensional row vectors. Here $\tilde{B}_t^f = B_t^f X_t$ and $\tilde{S}_t^f = S_t^f X_t$. Consider an exchange option with payoff $\max(K\tilde{S}_T^f - S_T^d, 0)$ in domestic currency at time T.

(a) Use the change-of-numeraire technique to show that the price at time $t = 0$ of the exchange option can be written as

$$KX_0 S_0^f \tilde{Q}^f(Z_T \geq 1/K) - S_0^d Q^d(Z_T \geq 1/K),$$

where $\tilde{Q}^f$ refers to the martingale measure with $\tilde{S}^f$ as numeraire, Q^d is the martingale measure with S^d as numeraire, and $Z_t = \tilde{S}_t^f / S_t^d$. (7 p)

(b) Show that the price in (a) can be computed as

$$KX_0 S_0^f N(d_1) - S_0^d N(d_2),$$

and determine d_1 and d_2. Here $N(x)$ is the distribution function of a standard normal distribution. (3 p)