KTH Matematik

EXAMINATION IN SF2975 FINANCIAL DERIVATIVES, 2010-08-26, 08:00–13:00.

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Allowed technical aids: none.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

Bonus points: $10-12$ bonus point give full credit for Problem 1. $7-9$ bonus point give full credit for Problem 1a and 1b. $4-6$ bonus points give full credit for Problem 1a.

Good luck!

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Problem 1

(a) Consider a two period binomial model where the stock price evolves according to Figure 1 and the probability of an up-move is $p = 0.6$ under the original P-measure. The interest rate in each period is constant equal to $r = 0$. Compute the price of an up-and-out put option (barrier option) with barrier $L = 105$ and strike $K = 105$. (4 p)

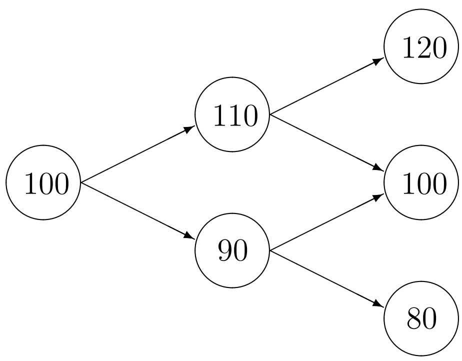

Figure 1: Binomial tree in Problem 1a.

(b) Consider the following two options portfolios where all options mature at time T. Let $K_1 < K_2 < K_3$ be strikes with $K_2 - K_1 = K_3 - K_2 = \Delta > 0$. Portfolio A consists of

- 1 long put with strike K_1,
- 1 long call with strike K_1,
- 1 short call with strike K_2,
- 1 short put with strike K_2,
- 1 long put with strike K_3,
- 1 long call with strike K_3.

Portfolio B consists of

- 1 long put with strike K_2,
- 1 long call with strike K_2,
- a debt of amount Δ maturing at T.

Which of the two portfolios is more expensive? (3 p)

(c) Consider a multidimensional Black-Scholes model with d stocks and a deterministic bank account. State the First Fundamental Theorem (of Mathematical Finance). (3 p)

Problem 2

Consider the Black-Scholes model where

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma S(t)dW(t), \quad S(0) = s, \\ dB(t) &= rB(t)dt, \quad B(0) = 1. \end{aligned}$$

Let $c(s, T, K)$ be the price of a call option with spot price s, time to maturity T, and strike K. That is, $C(t) = c(S(t), T - t, K)$ is the price at time t of a European call option with maturity T and strike K. Consider an at-the-money call-option on the call-option. That is, a contract X with maturity $T_0 < T$ and strike K_0 that gives the holder the option to buy at time T_0 for the price K_0 a European call option with maturity T and strike S_{T_0}. The payoff of X at time T_0 is given by $\max\{c(S_{T_0}, T - T_0, S_{T_0}) - K_0, 0\}$.
Determine the price of X at time $t < T_0$. (10 p)

Problem 3

The change-of-numeraire technique is useful for computing prices of contingent claims. For an arbitrage free market with traded assets $S^{(0)}, \dots, S^{(d)}$ the price at time t of a T-claim X is given by

$$\Pi(t; X) = S_t^{(i)} E^{Q^i}\Big[\frac{X}{S_T^{(i)}} \mid \mathcal{F}_t\Big],$$

where Q^i is the martingale measure with $S^{(i)}$ as numeraire. Changing the numeraire, say from $S^{(0)}$ to $S^{(1)}$, can be very helpful if the expectation $E^{Q^0}\left[\frac{X}{S_T^{(0)}} \mid \mathcal{F}_t\right]$ is difficult to compute but $E^{Q^1}\left[\frac{X}{S_T^{(1)}} \mid \mathcal{F}_t\right]$ is easy.
Give an example (non-trivial) of your own choice where the change-of-numeraire technique is useful to compute the price of a contingent claim. (10 p)

Problem 4

The *Term Structure Equation* is a PDE (partial differential equation) for the arbitrage free price $F^T(t,r)$ of a zero-coupon bond with maturity T (T-bond) as a function of time $t \in [0,T]$ and the short rate r.
Suppose the short rate is given by

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t), \quad r(0) = r_0,$$

where W is a Brownian motion under P. Suppose every T-bond has price at time t of the form $F^T(t, r(t))$, $t < T$, and that $F^T(t,r)$ is a sufficiently smooth function. Suppose that an equivalent martingale measure Q exists under which $F^T(t, r(t))/B(t)$ is a martingale. Here $B(t) = \exp\{\int_0^t r(s)ds\}$ is a bank account.
Derive the Term Structure Equation. The term structure equation must be expressed with $\mu(t,r)$, $\sigma(t,r)$ and the Girsanov kernel $\varphi(t)$ that defines the Girsanov transformation from P to Q. (10 p)

Problem 5

Consider a market with a bank account $dB(t) = rB(t)dt$, $B(0) = 1$, and two stocks

$$dS^{(1)}(t) = \mu^{(1)}S^{(1)}dt + S^{(1)}(t)\sigma_1 dW(t), \; S_0^{(1)} = s_1,$$
$$dS^{(2)}(t) = \mu^{(2)}S^{(2)}dt + S^{(2)}(t)\sigma_2 dW(t), \; S_0^{(2)} = s_2.$$

Here $\sigma_1 = (\sigma_{11} \; \sigma_{12})$, $\sigma_2 = (\sigma_{21} \; \sigma_{22})$ are such that the matrix

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

is invertible, and $W(t)$ is a two-dimensional Brownian motion. Suppose the stocks pays continuous dividends where the dividend processes are given by

$$dD^{(1)}(t) = \delta^{(1)}S^{(1)}(t)dt,$$
$$dD^{(2)}(t) = \delta^{(2)}S^{(2)}(t)dt,$$

with $\delta^{(1)}, \delta^{(2)} > 0$.
(a) Derive the Q-dynamics of $S^{(1)}$ and $S^{(2)}$. (5 p)

(b) Consider a contract with maturity T where you, at time T, receive the amount $D_T^{(1)} - \alpha D_T^{(2)}$, where α is a positive number. If this amount is positive you receive money and if it is negative you have to pay. You can view this contract as a swap of dividends. That is, you receive the dividends from one share of $S^{(1)}$ and have to pay the dividends from α number of $S^{(2)}$ shares. Determine α such that the price of the contract is 0 at time 0. (5 p)