Solution to Examination in SF2975 Financial derivatives 2012-03-13

Problem 1

(a) The risk-neutral probabilities are given by

$$q_u = \frac{1.05 - 0.8}{1.3 - 0.8} = 0.5 \text{ and } q_d = \frac{1.3 - 1.05}{1.3 - 0.8} = 0.5.$$

The payoff at the final nodes are

$$\max(169 - 148, 0) = 21$$
$$\max(104 - 148, 0) = 0$$
$$\max(64 - 148, 0) = 0$$

The prices at time 1 are

$$(0.5 \cdot 21 + 0.5 \cdot 0)/1.05 = 10$$
$$(0.5 \cdot 0 + 0.5 \cdot 0)/1.05 = 0$$

Finally, the price at time 0 is equal to

$$\frac{0.5 \cdot 10 + 0.5 \cdot 0}{1.05} = \frac{100}{21} \approx 4.76.$$

(b) See proposition 9.2 in Björk.

(c) We use the Feynman-Kac formula, which says that the solution F to the PDE has the stochastic representation

$$F(t, x) = E_{t,x}\left[X(T)^2\right],$$

where the stochastic process X evolves according to

$$dX(u) = adu + bdW(u); \ X(t) = x,$$

where W is a Wiener process. Hence

$$X(T) = X(t) + a(T - t) + b(W(T) - W(t)),$$

and this yields

$$X(T)|X(t) = x \sim N(x + a(T - t), b\sqrt{T - t}).$$

We get

$$\begin{aligned} F(t, x) &= E_{t,x}\left[X(T)^2\right] \\ &= (E_{t,x}[X(T)])^2 + Var_{t,x}(X(T)) \\ &= (x + a(T - t))^2 + b^2(T - t) \\ &= x^2 + (2ax + b^2)(T - t) + a^2(T - t)^2. \end{aligned}$$

Problem 2

The price of X at time $t \in [0, T]$ is given by the risk-neutral valuation formula

$$\Pi(t; X) = e^{-r(T-t)} E^Q \left[\max(\ln S(T), 0) | \mathcal{F}_t \right],$$

where $(\mathcal{F}_t)$ is the filtration generated by S. Under Q the stock price has dynamics

$$dS(t) = (r - \delta) S(t) dt + \sigma S(t) dW^Q(t),$$

where W^Q is a Q-Wiener process. The solution to this SDE is

$$S(T) = S(t) e^{\left(r - \delta - \sigma^2/2\right)(T-t) + \sigma(W^Q(T) - W^Q(t))},$$

and hence

$$\ln S(T) = \ln S(t) + (r - \delta - \sigma^2/2)(T - t) + \sigma(W^Q(T) - W^Q(t)).$$

We get

$$\begin{aligned}
\Pi(t; X) &= e^{-r(T-t)} E^Q \left[\max(\ln S(t) + (r - \delta - \sigma^2/2)(T - t) + \sigma(W^Q(T) - W^Q(t)), 0) | \mathcal{F}_t \right] \\
&= e^{-r(T-t)} \int_{-\infty}^{\infty} \max(\ln S(t) + (r - \delta - \sigma^2/2)(T - t) + \sigma z \sqrt{T - t}, 0) \varphi(z) dz \\
&= e^{-r(T-t)} \int_{z_0}^{\infty} (\ln S(t) + (r - \delta - \sigma^2/2)(T - t) + \sigma z \sqrt{T - t}) \varphi(z) dz \\
&= e^{-r(T-t)} \left[\left(\ln S(t) + (r - \delta - \sigma^2/2)(T - t) \right) (1 - N(z_0)) + \sigma \sqrt{T - t} \int_{z_0}^{\infty} z \varphi(z) dz \right],
\end{aligned}$$

where

$$z_0 = -\frac{\ln S(t)}{\sigma \sqrt{T - t)}} - \frac{r - \delta - \sigma^2/2}{\sigma} \sqrt{T - t}.$$

Here we have used that under Q

$$W^Q(T) - W^Q(t) | \mathcal{F}_t \sim N(0, \sqrt{T - t})$$

and the notation

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \text{ and } N(z) = \int_{-\infty}^{z} \varphi(x) dx.$$

Using the Hint we finally get

$$\begin{aligned}
\Pi(t; X) &= e^{-r(T-t)} \left[\left(\ln S(t) + (r - \delta - \sigma^2/2)(T - t) \right) (1 - N(z_0)) + \sigma \sqrt{T - t} \varphi(z_0) \right] \\
&= e^{-r(T-t)} \left[\left(\ln S(t) + (r - \delta - \sigma^2/2)(T - t) \right) N(-z_0) + \sigma \sqrt{T - t} \varphi(-z_0) \right] \\
&= e^{-r(T-t)} \left[\left(\ln S(t) + (r - \delta - \sigma^2/2)(T - t) \right) N\left(\frac{\ln S(t)}{\sigma \sqrt{T - t}} + \frac{r - \delta - \sigma^2/2}{\sigma} \sqrt{T - t} \right) \right. \\
&\quad \left. + \sigma \sqrt{T - t} \varphi\left(\frac{\ln S(t)}{\sigma \sqrt{T - t}} + \frac{r - \delta - \sigma^2/2}{\sigma} \sqrt{T - t} \right) \right]
\end{aligned}$$

Problem 3

(a) Here we have an ATS model, and we know that in this case

$$p(t,T) = e^{A(t,T)-B(t,T)r(t)},$$

where A and B solves the equations

$$\begin{cases} \frac{\partial A}{\partial t}(t,T) &= bB(t,T) - \frac{1}{2}\sigma^2 B^2(t,T) \\ A(T,T) &= 0 \end{cases} \qquad \begin{cases} \frac{\partial B}{\partial t}(t,T) &= -1 \\ B(T,T) &= 0. \end{cases}$$

Solving the last equation yields

$$B(t,T) = T - t,$$

and inserting this in the first equation gives

$$\begin{aligned} A(T,T) - A(t,T) &= b\int_t^T (T-u)du - \frac{1}{2}\sigma^2 \int_t^T (T-u)^2 du \\ &= \frac{b}{2}(T-t)^2 - \frac{\sigma^2}{6}(T-t)^3. \end{aligned}$$

Using $A(T,T) = 0$ we get

$$p(t,T) = e^{\frac{\sigma^2}{6}(T-t)^3 - \frac{b}{2}(T-t)^2 - (T-t)r(t)}.$$

Alternatively we can solve the SDE for r directly to get

$$r(u) = r(t) + b(u-t) + \sigma(W^Q(u) - W^Q(t)),$$

and

$$\int_t^T r(u)du = r(t)(T-t) + \frac{b}{2}(T-t)^2 + \sigma\int_t^T (W^Q(u) - W^Q(t))du.$$

Hence, using the risk-neutral valuation formula,

$$\begin{aligned} p(t,T) &= E^Q\left[e^{-\int_t^T r(u)du} \middle| \mathcal{F}_t \right] \\ &= e^{-r(t)(T-t) - \frac{b}{2}(T-t)^2} E^Q\left[e^{-\sigma\int_t^T (W^Q(u) - W^Q(t))du} \middle| \mathcal{F}_t \right]. \end{aligned}$$

Since

$$\int_t^T (W^Q(u) - W^Q(t))du = \int_t^T (T-u)dW^Q(u)$$

and under Q

$$\int_t^T (T-u)dW^Q(u) \Big| \mathcal{F}_t \sim N\left(0, \sqrt{\int_t^T (T-u)^2 du}\right) = N\left(0, \sqrt{\frac{(T-t)^3}{6}}\right)$$

we get

$$\begin{aligned} p(t,T) &= e^{-r(t)(T-t) - \frac{b}{2}(T-t)^2} E^Q\left[e^{-\sigma\int_t^T (T-u)dW^Q(u)} \middle| \mathcal{F}_t \right] \\ &= e^{-r(t)(T-t) - \frac{b}{2}(T-t)^2 + \frac{\sigma^2}{6}(T-t)^3}. \end{aligned}$$

(b) We know that

$$\Pi(t;X) = p(t,T)E^{Q^T}\left[r(T)|\mathcal{F}_t\right],$$

where Q^T is the T-forward measure, and that

$$f(t,T) = E^{Q^T}\left[r(T)|\mathcal{F}_t\right].$$

In the Merton model we have (use $p(t,T)$ from (a))

$$f(t,T) = -\frac{\partial \ln p(t,T)}{\partial T} = r(t) + b(T-t) - \frac{\sigma^2}{2}(T-t)^2,$$

and hence, again using $p(t,T)$ from (a),

$$\Pi(t;X) = \big(r(t) + b(T-t) - \frac{\sigma^2}{2}(T-t)^2\big)e^{\frac{\sigma^2}{6}(T-t)^3 - \frac{b}{2}(T-t)^2 - (T-t)r(t)}.$$

Problem 4

(a) A self-financing portfolio $h(t) = (h^B(t), h^S(t))$ has dynamics

$$dV^h(t) = h^B(t)dB(t) + h^S(t)dS(t),$$

so we get

$$\begin{aligned} dV^h(t) &= (1-u_0)V^h(t)\frac{dB(t)}{B(t)} + u_0V^h(t)\frac{dS(t)}{S(t)} \\ &= V^h(t)\left[((1-u_0)r + u_0\alpha)dt + u_0\sigma dW(t)\right] \\ &= V^h(t)\left[(r + u_0(\alpha - r))dt + u_0\sigma dW(t)\right] \end{aligned}$$

(b) The differential of $S(t)/V^h(t)$ is given by

$$d\left(\frac{S(t)}{V^h(t)}\right) = \frac{dS(t)}{V^h(t)} + S(t)d\left(\frac{1}{V^h(t)}\right) + dS(t)d\left(\frac{1}{V^h(t)}\right).$$

Now

$$\begin{aligned} d\left(\frac{1}{V^h(t)}\right) &= -\frac{dV^h(t)}{(V^h(t))^2} + \frac{(dV^h(t))^2}{(V^h(t))^3} \\ &= \frac{1}{V^h(t)}\left[(-r - u_0(\alpha - r) + u_0^2\sigma^2)dt - u_0\sigma dW(t)\right], \end{aligned}$$

and we get

$$d\left(\frac{S(t)}{V^h(t)}\right) = \frac{S(t)}{V^h(t)}\left[(\alpha - r - u_0(\alpha - r) + u_0^2\sigma^2 - u_0\sigma^2)dt + (\sigma - u_0\sigma)dW(t)\right]$$

With $u_0 = (\alpha - r)/\sigma^2$ we get the drift rate

$$\alpha - r - u_0(\alpha - r) + u_0^2\sigma^2 - u_0\sigma^2 = \alpha - r - \frac{(\alpha - r)^2}{\sigma^2} + \frac{(\alpha - r)^2}{\sigma^2} - (\alpha - r) = 0,$$

so $S(t)/V^h(t)$ is a P-martingale. The dynamics of $B(t)/V^h(t)$ are given by

$$d\left(\frac{B(t)}{V^h(t)}\right) = \frac{dB(t)}{V^h(t)} + B(t)d\left(\frac{1}{V^h(t)}\right),$$

and from this

$$d\left(\frac{B(t)}{V^h(t)}\right) = \frac{B(t)}{V^h(t)}\left[(r - r - u_0(\alpha - r) + u_0^2\sigma^2)dt - u_0\sigma dW(t)\right].$$

Inserting $u_0 = (\alpha - r)/\sigma^2$ we see that the drift rate is

$$-\frac{(\alpha - r)^2}{\sigma^2} + \frac{(\alpha - r)^2}{\sigma^2} = 0,$$

and also $B(t)/V^h(t)$ is a P-martingale.

(c) The likelihood process is given by

$$L(t) = \frac{B(0)}{V^h(0)} \cdot \frac{V^h(t)}{B(t)} = \frac{V^h(t)}{B(t)}, \ 0 \le t \le T$$

(see Proposition 26.4 in Björk).

Problem 5

(a) We know that

$$\Pi(t; X) = e^{-r(T-t)}E^Q\left[\sqrt{S_1(T)S_2(T)}\,\middle|\mathcal{F}_t\right],$$

where Q is the martingale measure where B is the numeraire. The drift rate under Q is equal to r for both S_1 and S_2. Itô's formula on $Z(t) := S_1(t)S_2(t)$ yields

$$\begin{aligned} dZ(t) &= S_1(t)dS_2(t) + S_2(t)dS_1(t) + dS_1(t)dS_2(t) \\ &= S_1(t)S_2(t)\Big[rdt + \sigma_{11}dW_1^Q(t) + \sigma_{12}dW_2^Q(t) \\ &\quad +rdt + \sigma_{21}dW_1^Q(t) + \sigma_{22}dW_2^Q(t) + \sigma_{11}\sigma_{21}dt + \sigma_{12}\sigma_{22}dt\Big] \\ &= Z(t)\Big[(2r + \sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22})dt + (\sigma_{11} + \sigma_{21})dW_1^Q(t) + (\sigma_{12} + \sigma_{22})dW_2^Q(t)\Big] \end{aligned}$$

Let

$$\alpha = 2r + \sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22},$$

and let

$$\sigma^2 = (\sigma_{11} + \sigma_{21})^2 + (\sigma_{12} + \sigma_{22})^2.$$

Then we can write

$$dZ(t) = Z(t)[\alpha dt + \sigma d\hat{W}^Q(t)],$$

where $\hat{W}^Q$ is a 1-dimensional Q-Wiener process. The solution to this SDE is

$$Z(T) = Z(t)e^{(\alpha - \sigma^2/2)(T-t) + \sigma(\hat{W}^Q(T) - \hat{W}^Q(t))},$$

and we get

$$\begin{aligned}
\Pi(t;X) &= e^{-r(T-t)}E^Q\left[\sqrt{Z(T)}\,\middle|\mathcal{F}_t\right] \\
&= e^{-r(T-t)}\sqrt{Z(t)}E^Q\left[e^{\frac{1}{2}(\alpha-\sigma^2/2)(T-t)+\frac{1}{2}\sigma(\hat{W}^Q(T)-\hat{W}^Q(t))}\,\middle|\mathcal{F}_t\right] \\
&= e^{-r(T-t)}\sqrt{Z(t)}e^{(\alpha/2-\sigma^2/4)(T-t)+(\sigma^2/8)(T-t)} \\
&= \left\{\text{use } Z(t)=S_1(t)S_2(t) \text{ and the expressions for } \alpha \text{ and } \sigma\right\} \\
&= \sqrt{S_1(t)S_2(t)}e^{-\frac{1}{8}[(\sigma_{11}-\sigma_{21})^2+(\sigma_{12}-\sigma_{22})^2]}.
\end{aligned}$$

Alternatively we can use the change of numeraire technique. Using $S_1(t)$ as numeraire we get

$$\Pi(t;X) = S_1(t)E^{Q_{S_1}}\left[\sqrt{\frac{S_2(T)}{S_1(T)}}\,\middle|\mathcal{F}_t\right].$$

The dynamics of $S_2(t)/S_1(t)$ under Q_{S_1} is

$$d\left(\frac{S_2(t)}{S_1(t)}\right) = \sigma\left(\frac{S_2(t)}{S_1(t)}\right)d\tilde{W}(t),$$

where σ is as above, and $\tilde{W}$ is a Q_{S_1}-Wiener process. Then

$$\frac{S_2(T)}{S_1(T)} = \frac{S_2(t)}{S_1(t)}e^{-(\sigma^2/2)(T-t)+\sigma(\tilde{W}(T)-\tilde{W}(t))},$$

and

$$\sqrt{\frac{S_2(T)}{S_1(T)}} = \sqrt{\frac{S_2(t)}{S_1(t)}}e^{-(\sigma^2/4)(T-t)+(\sigma/2)(\tilde{W}(T)-\tilde{W}(t))}.$$

Finally,

$$\begin{aligned}
\Pi(t;X) &= S_1(t)\sqrt{\frac{S_2(t)}{S_1(t)}}E^{Q_{S_1}}\left[e^{-(\sigma^2/4)(T-t)+(\sigma/2)(\tilde{W}(T)-\tilde{W}(t))}\,\middle|\mathcal{F}_t\right] \\
&= \sqrt{S_1(t)S_2(t)}e^{-\frac{1}{8}[(\sigma_{11}-\sigma_{21})^2+(\sigma_{12}-\sigma_{22})^2]}.
\end{aligned}$$