EXAMINATION IN SF2975 FINANCIAL DERIVATIVES, 2012-03-13, 14:00–19:00.

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Allowed technical aids: none.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

Bonus points: $10-12$ bonus point give full credit for Problem 1. $7-9$ bonus point give full credit for Problem 1a and 1b. $4-6$ bonus points give full credit for Problem 1a.

GOOD LUCK!

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Problem 1

(a) Consider the standard binomial model with two time steps as in Figure 1. The parameters are $s = 100$, $u = 1.3$, $d = 0.8$, $r = 0.05$, and the probability p of an up move is $p = 0.6$. Determine the price at time 0 of a European claim which at time 2 pays the amount

$$X = \max(S(2) - 148, 0).$$

(4 p)

(b) State and prove the put-call parity. (3 p)

(c) Solve the PDE

$$\begin{aligned} \frac{\partial F}{\partial t} + a\frac{\partial F}{\partial x} + \frac{b^2}{2}\frac{\partial^2 F}{\partial x^2} &= 0 \\ F(T, x) &= x^2 \end{aligned}$$

on $[0, T]$. Here $a, b \in \mathbb{R}$. (3 p)

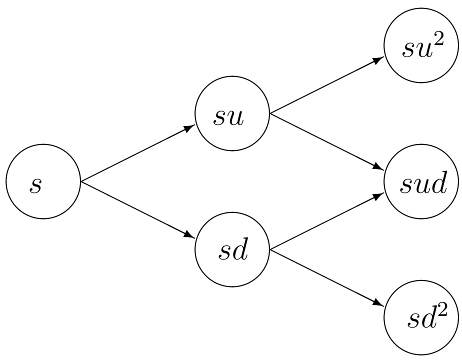

Figure 1: Binomial tree in Problem 1a.

Problem 2

Consider the standard Black–Scholes model with bank account dynamics

$$dB(t) = rB(t)dt;\ B(0) = 1$$

and stock price dynamics

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t);\ S(0) = s > 0.$$

Here $r \geq 0$, $\alpha \in \mathbb{R}$, $\sigma > 0$ and W is a 1-dimensional Wiener process under the objective measure P. Furthermore the stock pays dividends with a constant dividend yield $\delta \geq 0$. Determine for every $t \in [0, T]$ the arbitrage free price $\Pi(t; X)$ of the claim which pays

$$X = \max(\ln S(T), 0)$$

at time T.

Hint: It holds that $\int_a^\infty x\varphi(x)dx = \varphi(a)$ for every $a \in \mathbb{R}$, where φ is the density function of an $N(0, 1)$-distributed random variable. (10 p)

Problem 3

Consider the Merton model for the short rate under an equivalent martingale measure Q with the bank account as numeraire,

$$dr(t) = bdt + \sigma dW^Q(t).$$

Here $b \in \mathbb{R}$, $\sigma > 0$ and W^Q is a 1-dimensional Wiener process under Q.

(a) Determine for every $0 \leq t \leq T < \infty$ the arbitrage free price $p(t, T)$ for a zero coupon bond maturing at T. (5 p)

(b) Determine for every $t \in [0, T]$ the arbitrage free price $\Pi(t; X)$ of the claim which pays

$$X = r(T)$$

at time T. (5 p)

Problem 4

Again consider the standard Black–Scholes model described in Exercise 2 above, but now with $\delta = 0$ (i.e. the stock pays no dividends). A self-financing portfolio h is constructed according to the following. Take $u_0 \in \mathbb{R}$ and let

$$h^S(t) = u_0 \frac{V^h(t)}{S(t)} \text{ and } h^B(t) = (1 - u_0)\frac{V^h(t)}{B(t)}.$$

We also set $V^h(0) = 1$.

(a) Show that the dynamics of V^h are given by

$$dV^h(t) = V^h(t)\left[(r + u_0(\alpha - r))dt + u_0\sigma dW(t)\right].$$

(2 p)

(b) Show that the choice $u_0 = (\alpha - r)/\sigma^2$ makes both $B(t)/V^h(t)$ and $S(t)/V^h(t)$ martingales under P. (6 p)

(c) The result in (b) shows that using $V^h(t)$, with $u_0 = (\alpha - r)/\sigma^2$, as numeraire, the original measure P is also the pricing measure. What does the likelihood process occuring when we change measure from Q with B as numeraire to P with V^h as numeraire on $\mathcal{F}_T$ look like? (2 p)

Problem 5

Let the bank account process be given by

$$dB(t) = rB(t)dt; \; B(0) = 1$$

with $r \geq 0$.

Consider the model

$$\begin{aligned} dS_1(t) &= \alpha_1 S_1(t)dt + \sigma_{11} S_1(t)dW_1(t) + \sigma_{12} S_1(t)dW_2(t) \\ dS_2(t) &= \alpha_2 S_2(t)dt + \sigma_{21} S_2(t)dW_1(t) + \sigma_{22} S_2(t)dW_2(t). \end{aligned}$$

We assume $\alpha_1, \alpha_2 \in \mathbb{R}$, $S_1(0) = s_1 > 0$, $S_2(0) = s_2 > 0$ and that the matrix

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

is non-singular (i.e. its inverse exists). The process $(W_1(t), W_2(t))$ is a 2-dimensional Wiener process under the objective measure P. Determine for every $t \in [0, T]$ the arbitrage free price $\Pi(t; X)$ of the claim which pays

$$X = \sqrt{S_1(T)S_2(T)}$$

at time T. (10 p)