EXAMINATION IN SF2975 FINANCIAL DERIVATIVES, 2012-06-07, 14:00–19:00.

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Allowed technical aids: none.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

Bonus points: $10-12$ bonus point give full credit for Problem 1. $7-9$ bonus point give full credit for Problem 1a and 1b. $4-6$ bonus points give full credit for Problem 1a.

Good luck!

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Problem 1

(a) Consider the binomial model with two time steps as in Figure 1. The parameters are $s = 50$, $u = 1.2$, $d = 0.8$, $r = 0$, and the probability p of an up move is $p = 0.6$. Determine the price at time 0 of the lookback call which at time 2 pays the amount

$$X = S(2) - \min_{t \in \{0,1,2\}} S(t).$$

(4 p)

(b) Solve the PDE

$$\begin{aligned} \frac{\partial F}{\partial t} + b\frac{\partial^2 F}{\partial x^2} + aF &= x \\ F(T,x) &= 0 \end{aligned}$$

on $[0,T]$ where $a > 0$ and $b \geq 0$. (3 p)

(c) Assume that we have a short rate model such that there is an affine term structure (ATS). Show that the price at time t of a zero coupon bond maturing at T $(0 \leq t \leq T)$ is convex as a function of the short rate at time t. (3 p)

Problem 2

Consider the standard Black–Scholes model with bank account dynamics

$$dB(t) = rB(t)dt;\ B(0) = 1$$

and stock price dynamics

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t);\ S(0) = s > 0.$$

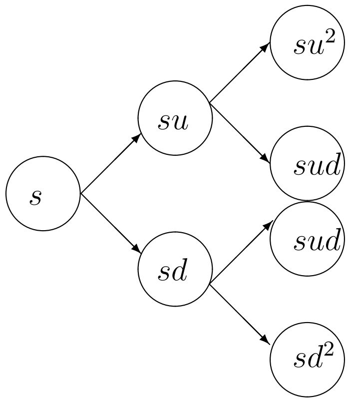

Figure 1: Binomial tree in Problem 1a.

Here $r \geq 0$, $\alpha \in \mathbb{R}$, $\sigma > 0$ and W is a 1-dimensional Wiener process under the objective measure P. The stock does not pay any dividends.

(a) Determine for every $t \in [0, T]$ the arbitrage free price $\Pi(t; X)$ of the claim which pays

$$X = S(T)^{\beta}$$

at time T, where $\beta \geq 0$. (5 p)

(b) Determine for every $t \in [0, T]$ the replicating portfolio $(h^B(t), h^S(t))$ of the contract in (a). (5 p)

Problem 3

In the Heath–Jarrow–Morton (HJM) approach the forward rates are modelled according to

$$df(t, T) = \alpha(t, T)dt + \sigma(t, T)dW(t),$$

where in this problem W is a 2-dimensional Wiener process under the objective measure P. We assume that the market is free of arbitrage. Determine the dynamics of $f(t, T)$ and the bond price dynamics $p(t, T)$ for $0 \leq t \leq T < \infty$ under the martingale measure Q when σ is given by the 1×2–dimensional vector

$$\sigma(t, T) = \left[\sigma_0 e^{-a(T-t)} \; \sigma_1(T - t)\right],$$

where $\sigma_0, \sigma_1, a > 0$. (10 p)

Problem 4

Again consider the standard Black–Scholes model described in Exercise 2 above. A self-financing portfolio h is constructed according to the following. Take $u_0 \in \mathbb{R}$ and let

$$h^S(t) = u_0 \frac{V^h(t)}{S(t)} \text{ and } h^B(t) = (1 - u_0)\frac{V^h(t)}{B(t)}.$$

We also set $V^h(0) = 1$.

(a) For every $t \in [0, T]$, find the arbitrage free price $\Pi(t; X)$ of the claim that pays

$$X = \ln V^h(T)$$

at time T. (5 p)

(b) Find the value of u_0 that maximizes

$$E[\ln(V^h(T))].$$

(5 p)

Problem 5

A market consists of three assets: a domestic bank account with deterministic rate r_d, a foregin bank account with deteterminisitc rate r_f and a stock denoted in the domenstic currency with dynamics

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t),$$

where W is a 1-dimensional Wiener process under the objective measure P. The spot exchange rate X has, under P, dynamics

$$dX(t) = \alpha_X X(t)dt + \sigma_X X(t)dW(t),$$

where W is the same 1-dimensional Wiener process as in the dynamics of the stock price. We assume that the market is free of arbitrage.

(a) Show that absence of arbitrage implies that we must have

$$\alpha_X = r_d - r_f + \frac{\sigma_X}{\sigma}(\alpha - r_d).$$

(4 p)

(b) Determine for every $t \in [0, T]$ the arbitrage free price $\Pi(t; Z)$ of the claim which in domestic currency pays

$$Z = \mathbf{1}(X(T) \geq K) = \begin{cases} 1 & \text{if} \quad X(T) \geq K \\ 0 & \text{if} \quad X(T) < K \end{cases}$$

at time T. (6 p)