KTH Mathematics

Exam in SF2975 Financial Derivatives.
Friday March 15 2013 14.00-19.00.

---

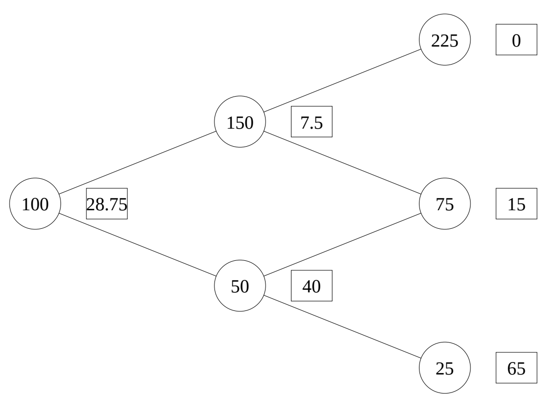
Answers and suggestions for solutions.

---

**1.** (a) For the martingale probabilities we have

$$q = \frac{1+r-d}{u-d} = 0.5$$

Using them we obtain the following binomial tree where the value of the stock is written in the nodes, and the value of the option is written in the adjacent boxes. The value 20 adjacent to the node with stock price 80 is obtained as $\max\{100-80, \frac{1}{1.1}(0.5 \cdot 0 + 0.5 \cdot 36)\}$. Thus, an early exercise of the option is optimal at this node!




The price of the put option is thus 9.09 kr.

(b) i. $\Delta = \frac{\partial P(t,s)}{\partial s}$.

ii. A portfolio is delta neutral if it has $\Delta = 0$. If we let $x_S$ denote the number of underlying stocks you should buy, and use that $\Delta = 1$ for the underlying stock itself, we obtain the following equation for $x_S$

$$1.3 + x_S \cdot 1 = 0.$$

You should therefore sell 1.3 stocks.

iii. A high value of $\Gamma = \frac{\partial^2 P(t,s)}{\partial s^2} = \frac{\partial \Delta}{\partial s}$ indicates that $\Delta$ is very sensitive to changes in the underlying stock price, which means that you will probably have to rebalance your portfolio quite frequently (as long as $\Gamma$ remains high) in order to keep it delta neutral.

(c) i. **Theorem 1 (First Fundamental Theorem)** *The model is arbitrage free essentially if and only if there exists a (local) martingale measure $Q$.*

ii. **Theorem 2 (Second Fundamental Theorem)** *Assume that the market is arbitrage free. Then the market is complete if and only if the martingale measure is unique.*

**2.** (a) We have the following equation

$$0 = \Pi(t; X) = e^{-r(T-t)} E^Q \left[ S_T - f(t; T, S_T) \middle| \mathcal{F}_t \right].$$

Solving for the forward price we obtain (use that $f(t; T, S_T) \in \mathcal{F}_t$)

$$f(t; T, S_T) = E^Q[S_T | \mathcal{F}_t].$$

Since S/B is a $Q$-martingale and $B_t = e^{rt}$ we have that

$$f(t; T, S_T) = B_T E^Q \left[ \frac{S_T}{B_T} \middle| \mathcal{F}_t \right] = B_T \frac{S_t}{B_t} = e^{r(T-t)} S_t.$$

(b) The payoff of the break forward can be written as

$$X = \max\{S_T, \underbrace{S_0 e^{rT}}_{const.}\} - K = \max\{S_T - S_0 e^{rT}, 0\} + S_0 e^{rT} - K.$$

the price is therefore given by

$$\begin{aligned} \Pi(t; X) &= e^{-r(T-t)} E^Q[\max\{S_T - S_0 e^{rT}, 0\} + S_0 e^{rT} - K | \mathcal{F}_t] \\ &= C(t, S_t, S_0 e^{rT}, T, r, \sigma) + e^{-r(T-t)}(S_0 e^{rT} - K), \end{aligned}$$

where $C(t, s, K, T, r, \sigma)$ denotes the standard Black-Scholes price at time $t$ of a European call option with exercise price $K$ and expiry date $T$, when the current price of the underlying is $s$, the interest rate is $r$, and the volatility of the underlying is $\sigma$. For future use we let $P(t, s, K, T, r, \sigma)$ denote the price of the corresponding put option. Setting the price equal to zero at time $t = 0$ we obtain the following expression for the delivery price $K$

$$\begin{aligned} K &= S_0 e^{rT} + e^{rT} C(0, S_0, S_0 e^{rT}, T, r, \sigma) \\ &= S_0 e^{rT}(1 + \Phi[d_1(0, S_0)] - \Phi[d_2(0, S_0)]). \end{aligned}$$

(c) The payoff $X$ at time $T_0$ of the exchange option is given by

$$X = \max\{C(T_0, S_{T_0}, K, T, r, \sigma) - P(T_0, S_{T_0}, K, T, r, \sigma), 0\}.$$

Using put-call-parity $P(t, s, K, T, r, \sigma) = K e^{-r(T-t)} + C(t, s, K, T, r, \sigma) - s$ this payoff can be written as

$$X = \max\{S_{T_0} - K e^{-r(T-T_0)}, 0\}.$$

We now see that the price of the exchange option is given by

$$\begin{aligned} \Pi(t; X) &= e^{-r(T_0-t)} E^Q \left[ \max\{S_{T_0} - K e^{-r(T-T_0)}, 0\} \middle| \mathcal{F}_t \right] \\ &= C\left(t, S_t, K e^{-r(T-T_0)}, T_0, r, \sigma\right), \end{aligned}$$

which can be computed using the Black-Scholes formula.

**3.** (a) Recall that

$$p(t,T) = e^{-\int_t^T f(t,u)du}.$$

We thus have that

$$p(t,T) = e^{Z(t)},$$

where

$$Z(t) = -\int_t^T f(t,u)du.$$

Using one of the hints we get

$$\begin{aligned} dZ(t) &= f(t,t)dt - \int_t^T df(t,u)du \\ &= r(t)dt - \int_t^T \left[\alpha(t,u)dt + \sigma(t,u)dW_t\right] du \\ &= \left[r(t) - \int_t^T \alpha(t,u)du\right] dt - \left[\int_t^T \sigma(t,u)du\right] dW_t \end{aligned}$$

Finally using Itô's formula on $p(t,T) = \exp\{Z(t)\}$ we get

$$dp(t,T) = \{r(t) + b(t,T)\}p(t,T)dt + a(t,T)p(t,T)dW(t),$$

where

$$\begin{cases} a(t,T) &= -\int_t^T \sigma(t,u)du, \\ b(t,T) &= -\int_t^T \alpha(t,u)du + \frac{1}{2}a^2(t,T). \end{cases}$$

(b) The drift of any price process under the risk neutral martingale measure is equal to the short rate. This means that we have $b(t,T) = 0$. Using the expression for $b(t,T)$ and taking the derivative w.r.t. $T$ we get the HJM drift condition

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)du.$$

(c) Using the HJM drift condition we obtain

$$\alpha(t,T) = \sigma^2 e^{-a(T-t)} \int_t^T e^{-a(u-t)}du = -\frac{\sigma^2}{a}e^{-a(T-t)}(e^{-a(T-t)} - 1).$$

Thus the forward rates rates are given by

$$f(t,T) = f(0,T) - \int_0^t \frac{\sigma^2}{a}e^{-a(T-u)}(e^{-a(T-u)} - 1)du + \int_0^t \sigma e^{-a(T-u)}dV_u$$

where $V$ denotes a $Q$-wiener process. Since $r(t) = f(t,t)$ we have that

$$r(t) = f(0,t) - \int_0^t \frac{\sigma^2}{a}e^{-a(t-u)}(e^{-a(t-u)} - 1)du + \int_0^t \sigma e^{-a(t-u)}dV_u.$$

Note that Wiener integral can be written as

$$\int_0^t \sigma e^{-a(t-u)}dV_u = e^{-at}\int_0^t \sigma e^{au}dV_u = e^{-at}Z_t,$$

if we define $Z_t = \int_0^t \sigma e^{au} dV_u$. Thus we have that

$$r(t) = m(t) + e^{-at} Z_t, \tag{1}$$

with

$$\begin{aligned} m(t) &= f(0,t) - \int_0^t \frac{\sigma^2}{a} e^{-a(t-u)} (e^{-a(t-u)} - 1) du & (2) \\ &= f(0,t) + \frac{\sigma^2}{2a^2} \left(1 - e^{-at}\right)^2 . & (3) \end{aligned}$$

Applying Itô's formula to (1) we obtain the following dynamics for $r$ (use that $dZ = \sigma e^{at} dV_t$)

$$\begin{aligned} dr_t &= \left(m'(t) - ae^{-at} Z_t\right) dt + e^{-at} dZ_t \\ &= \left(m'(t) - ae^{-at} Z_t\right) dt + e^{-at} \sigma e^{at} dV_t \\ &= \left[m'(t) + am(t) - a\left(m(t) + e^{-at} Z_t\right)\right] dt + \sigma dV_t \\ &= (m'(t) + am(t) - ar_t) dt + \sigma dV_t. \end{aligned}$$

Thus $r$ has the required form with $\phi(t) = m'(t) + am(t)$.

(d) Since a forward rate model is automatically fitted to the initial yield curve, we see that we should choose $\phi(t) = m'(t) + am(t)$, where $m$ is given by (2) with $f(0,t)$ equal to the observed forward rate curve.

**4.** (a) Under $Q^d$ the following processes should be martingales

$$\frac{B^d}{B^d}, \quad \frac{\widetilde{B}^f}{B^d} = \frac{XB^f}{B^d}, \quad \frac{\widetilde{S}^f}{B^d} = \frac{XS^f}{B^d}.$$

This means that the processes $B^d$, $\widetilde{B}^f$, and $\widetilde{S}^f$ should have a local rate of return equal to $r_d$. Perform a Girsanov transformation according to

$$dQ^d = L^d(t) dP \qquad \text{on } \mathcal{F}_t,$$

where

$$\begin{cases} dL_t^d = L_t^d \varphi^* d\overline{W}_t, \\ L_0^d = 1. \end{cases}$$

Here * denotes transpose. Using Itô's formula and Girsanov's theorem we find the $Q^d$-dynamics of the above processes to be

$$\begin{aligned} d\widetilde{B}^f &= \widetilde{B}^f (r_f + \alpha_X + \sigma_X \varphi) dt + \widetilde{B}^f \sigma_X d\overline{V}, \\ d\widetilde{S}^f &= \widetilde{S}^f (\alpha_f + \alpha_X + \sigma_f \sigma_X^* + [\sigma_f + \sigma_X]\varphi) dt + \widetilde{S}^f (\sigma_f + \sigma_X) d\overline{V}. \end{aligned}$$

Here $\overline{V} = (V_1, V_2)^*$ is the two dimensional $Q^d$-Wiener process, defined by $d\overline{V} = d\overline{W} - \varphi dt$. Setting the local rates of return equal to $r_d$ gives the following system of equations

$$\begin{aligned} \sigma_X \varphi &= r_d - r_f - \alpha_X, \\ (\sigma_f + \sigma_X)\varphi &= r_d - \alpha_f - \alpha_X - \sigma_f \sigma_X^*. \end{aligned}$$

Thus, under $Q^d$ we have

$$\begin{aligned} dS^f &= (\alpha_f + \sigma_f \varphi) S^f dt + S^f \sigma_f d\overline{V} \\ &= S^f (r_f - \sigma_f \sigma_X^*) dt + S^f \sigma_f d\overline{V}. \end{aligned}$$

(b) The price of the quanto forward contract is given by

$$\Pi(t,T) = e^{-r_d(T-t)} E^d \left[ S_T^f - K^f \middle| \mathcal{F}_t \right].$$

Setting this equal to zero and solving for the delivery price $K^f$ yields

$$K^f = E^d \left[ S_T^f | \mathcal{F}_t \right].$$

Now, we have that

$$S_u^f = S_t^f + \int_t^u (r_f - \sigma_f \sigma_X^*) S_\tau^f d\tau + \int_t^u S_\tau^f \sigma_f d\overline{V}_\tau.$$

Let $m(u) = E^d[S_u^f | \mathcal{F}_t]$ and take the conditional expectation with respect to $\mathcal{F}_t$ to obtain

$$m(u) = S_t^f + \int_t^u (r_f - \sigma_f \sigma_X^*) m(\tau) d\tau + 0.$$

This gives the ODE

$$\begin{cases} \dot{m} &= (r_f - \sigma_f \sigma_X^*) m, \\ m(t) &= S_t^f, \end{cases}$$

with solution $m(u) = S_t^f \exp\{(r_f - \sigma_f \sigma_X^*)(u-t)\}$. The correct delivery price is thus $K^f = S_t^f \exp\{(r_f - \sigma_f \sigma_X^*)(T-t)\}$.

**5.** (a) A portfolio is a vector process $\mathbf{h} = (h_0, h_1)$ which is adapted (really it should be predictable) and sufficiently integrable. Here $h_0$ is the number of risk free assets in the portfolio and $h_1$ is the number of stocks. The value process is given by

$$V(t; \mathbf{h}) = h_0(t) B(t) + h_1(t) S(t).$$

The gain process $G$ of the stock is defined by $G(t) = S(t) + D(t)$ and a portfolio $\mathbf{h}$ is self-financing if it holds that

$$dV(t; \mathbf{h}) = h_0(t) dB(t) + h_1(t) dG(t).$$

(b) A probability measure $Q \sim P$ is a martingale measure if the normalized gain process

$$G^Z(t) = \frac{S(t)}{B(t)} + \int_0^t \frac{1}{B(s)} dD(s),$$

is a $Q$-martingale.

(c) From Exercise (a) we know that the value process of a self-financing portfolio satisfies

$$dV(t; \mathbf{h}) = h_0(t) dB(t) + h_1(t) dG(t).$$

Now, defining the relative portfolio according to

$$u_0(t) = \frac{h_0(t)B(t)}{V(t)}, \qquad u_1 = \frac{h_1(t)S(t)}{V(t)},$$

we have that the value process of a self-financing relative portfolio should satisfy

$$dV(t;\mathbf{u}) = u_0(t)V(t)\frac{dB(t)}{B(t)} + u_1(t)V(t)\frac{dG(t)}{S(t)}.$$

Now, we are interested in the self-financing relative portfolio $\mathbf{u} = (0, 1)$, i.e. the relative portfolio which at all times has all its money invested in the risky asset. Recall that the dynamics under $Q$ of a risky asset paying a constant dividend yield of $\delta$ are given by

$$dS_t = (r - \delta)S_t dt + \sigma S_t dU_t,$$

where $U$ denotes a $Q$-Wiener process. The value process of this portfolio is then seen to solve

$$dV_t = 1 \cdot V_t \cdot \frac{dS_t + dD_t}{S_t} = rV_t dt + \sigma V_t dU_t, \tag{4}$$

$$V_0 = S_0, \tag{5}$$

where we have used that we should start out owning exactly one risky asset to begin with. Thus, the value process is given by geometrical Brownian motion, and the explicit solution for $V$ is given by

$$V_t = S_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma U_t\right\}.$$

Finding the number of stocks owned at $t$, $h_1(t)$, is now easy. Simply use that $V(t) = h_1(t)S(t)$ to obtain

$$h_1(t) = \frac{V(t)}{S(t)} = \frac{S_0}{S_t}\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma U_t\right\} = e^{\delta t}. \tag{6}$$

The number of risky assets owned at time $T$ is thus given by equation (6) evaluated at $t = T$, and the dynamics of the portfolio are given by equation (4).