KTH Mathematics

Exam in SF2975 Financial Derivatives.
Thursday May 30 2013 14.00-19.00.

Answers and suggestions for solutions.

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1. (a) According to the First Fundamental Theorem the model is free of arbitrage if and only if there exists a martingale measure. We thus need to prove that there exists a $0 \leq q \leq 1$ such that

$$s = \frac{1}{1+r}\left[q \cdot su + (1-q) \cdot sd\right].$$

Solving for q we obtain

$$q = \frac{(1+r) - d}{u - d}$$

which is well defined, and satisfies $0 \leq q \leq 1$ as long as $d < 1 + r < u$.

(b) Yes, it is possible to compute the price of the European call option given the information in the exercise. First use the price of the zero coupon bond to find the short rate r^* you should use by solving

$$p(t,T) = 0.95 = e^{-r^* \cdot 1}.$$

Then use that the price of an American call option equals the price of a European call option to find the implied volatility σ_{imp} from

$$c(0, 100, 100, 1, r^*, \sigma_{imp}) = 16.13.$$

Here $c(t, s, K, T, r, \sigma)$ denotes the standard Black-Scholes price at time t of a European call option with exercise price K and expiry date T, when the current price of the underlying is s, the interest rate is r, and the volatility of the underlying is σ. The price of the call option is now given by the Black-Scholes formula for $c(0, 100, 110, 0.5, r^*, \sigma_{imp})$.

(c) Let $P(t, s)$ denote the price of the portfolio at time t when $S_t = s$. We then have that

$$P(t, s) = c(t, s) - p(t, s).$$

Here $c(t, s)$ denotes the standard Black-Scholes price at time t of a European call option with exercise price K and expiry date T, when the current price of the underlying is s. The price of the corresponding put option is denoted by $p(t,s)$. By definition we have that

$$\Delta_P(t,s) = \frac{\partial P(t,s)}{\partial s}.$$

The computation of this derivative will simplify considerably if if we use put-call-parity, that is

$$p(t,s) = Ke^{-r(T-t)} + c(t,s) - s.$$

We then have that

$$P(t,s) = c(t,s) - \Big(Ke^{-r(T-t)} + c(t,s) - s\Big) = s - Ke^{-r(T-t)}.$$

the delta of the portfolio is thus

$$\Delta_P(t,s) = \frac{\partial P(t,s)}{\partial s} = 1.$$

2. (a) Denote the payoff function by ϕ and note that

$$\phi(S_T) = -a + \max\{S_T - x_1, 0\} - 2\max\{S_T - x_2, 0\} + \max\{S_T - x_3, 0\}.$$

The price of the contract is therefore

$$\begin{aligned}
\Pi_t &= e^{-r(T-t)}E^Q[-a + \max\{S_T - x_1, 0\} - 2\max\{S_T - x_2, 0\} \\
&\quad + \max\{S_T - x_3, 0\}|\mathcal{F}_t] \\
&= -e^{-r(T-t)}a + c(t, S_t, x_1, T, r, \sigma) - 2c(t, S_t, x_2, T, r, \sigma) \\
&\quad + c(t, S_t, x_3, T, r, \sigma) \\
&= -e^{-r(T-t)}a + c(t, S_t, 0.95S_t, T, r, \sigma) - 2c(t, S_t, S_t, T, r, \sigma) \\
&\quad + c(t, S_t, 1.05S_t, T, r, \sigma).
\end{aligned}$$

Here $c(t, s, K, T, r, \sigma)$ denotes the standard Black-Scholes price at time t of a European call option with exercise price K and expiry date T, when the current price of the underlying is s, the interest rate is r, and the volatility of the underlying is σ. If we want the price of the claim to be zero a should be chosen as

$$a = e^{r(T-t)}\left[c(t, S_t, 0.95S_t, T, r, \sigma) - 2c(t, S_t, S_t, T, r, \sigma) + c(t, S_t, 1.05S_t, T, r, \sigma)\right],$$

where $c(t, s, K, T, r, \sigma)$ is given by the Black-Scholes formula.

(b) The price of the claim is given by

$$\Pi_t = e^{-r(T-t)}E^Q\left[S_T^2 I_{\{S_T > K\}}\middle|\,\mathcal{F}_t\right].$$

Since $S_T = S_t e^Z$ where $Z \in N\left((r - \sigma^2/2)(T-t), \sigma^2(T-t)\right)$ this can be written as

$$\Pi_t = e^{-r(T-t)}\int_{\ln\left\{\frac{K}{S_t}\right\}}^{\infty} S_t^2 e^{2z}\varphi(z)dz,$$

where φ denotes the density of a $N\left((r - \sigma^2/2)(T-t), \sigma^2(T-t)\right)$-distribution. Now use that the density function for a $N(m, \sigma^2)$-distributed random variable is

$\varphi(z) = e^{-(z-m)^2/(2\sigma^2)}/(\sigma\sqrt{2\pi})$, and then complete the square in the exponent. This yields

$$\Pi_t = e^{-r(T-t)} e^{(2r+\sigma^2)(T-t)} S_t^2 \int_{\ln\left\{\frac{K}{S_t}\right\}}^{\infty} \psi(u)du,$$

where ψ denotes the density of a $N\left((r+3\sigma^2/2)(T-t), \sigma^2(T-t)\right)$-distribution. If we let U denote a $N\left((r+3\sigma^2/2)(T-t), \sigma^2(T-t)\right)$-distributed random variable. Then we have that

$$\begin{aligned}
\Pi_t &= e^{(r+\sigma^2)(T-t)} S_t^2 Q\left(U > \ln\left\{\frac{K}{S_t}\right\}\right) \\
&= e^{(r+\sigma^2)(T-t)} S_t^2 \left[1 - N\left(\frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left\{\frac{K}{S_t}\right\} - \left(r + \frac{3}{2}\sigma^2\right)(T-t)\right\}\right)\right] \\
&= e^{(r+\sigma^2)(T-t)} S_t^2 N\left(\frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left\{\frac{S_t}{K}\right\} + \left(r + \frac{3}{2}\sigma^2\right)(T-t)\right\}\right),
\end{aligned}$$

where we have used one of the hints to obtain the last equality.

3. (a) We have that

$$\begin{aligned}
V^h(t) &= h_B(t)B(t) + h_S(t)S(t) \\
&= [F(t,S(t)) - S(t)F_s(t,S(t))]B(t) + F_s(t,S(t))S(t) = F(t,S(t)),
\end{aligned}$$

since $B(t) \equiv 1$. That $V^h(T) = g(S(T))$ now follows from the boundary condition of PDE.

The self-financing condition is

$$dV^h(t) = h_B(t)dB(t) + h_S(t)dS(t).$$

As $dB(t) \equiv 0$ this reduces to

$$dV^h(t) = F_s(t,S(t))\sigma(t)S(t)dW^Q(t).$$

Since $V^h(t) = F(t,S(t))$ the Itô formula yields that

$$\begin{aligned}
dV_t^h &= F_t dt + F_s dS_t + \frac{1}{2}F_{ss}dS_t^2 \\
&= F_s(t,S(t))\sigma(t)S(t)dW^Q(t) + \frac{1}{2}\left[\sigma^2(t) - \sigma_{BS}^2\right] S^2(t)F_{ss}(t,S(t))
\end{aligned}$$

where the PDE for F has been used to substitute out F_t. Hence the strategy is self-financing precisely when

$$\frac{1}{2}\left[\sigma^2(t) - \sigma_{BS}^2\right] S^2(t)F_{ss}(t,S(t)) = 0.$$

If $\sigma(t) = \sigma_{BS}$ we are in the standard Black-Scholes setting (with zero interest rate) and the PDE for F is the PDE that gives arbitage-free prices for simple claims, and the strategy h is the standard delta hedge, which is self-financing in the standard setting.

(b) Yes, there will be arbitrage opportunities. This since the price of a contingent T-claim X in the model with stochastic interest rates is given by

$$\Pi_t^{model}[X] = E\left[e^{-\int_t^T r_s ds} X \,\middle|\, \mathcal{F}_t\right],$$

while the risk manager will compute the price according to the formula

$$\Pi_t^{manager}[X] = e^{r(t)(T-t)} E[X|\mathcal{F}_t].$$

In general these two prices will not be equal, and therefore arbitrage opportunities exist. Suppose there is a claim X for which $\Pi_0^{manager}[X] > \Pi_0^{model}[X]$. Then sell the claim to the manager (who will accept to buy at this price since he believes it to be correct) and buy the claim for $\Pi_t^{model}[X]$ from someone else on the market (since the stochastic interest rate model is assumed to be correct, there should be someone out there, who is willing to sell at this price). Invest the earnings from the transaction in the risk free asset. Your net investment at time 0 will be 0, and at time T you can pay the manager what you owe him using what you receive as an owner of the claim. But you will still have earnings in the bank! Thus you will have created an arbitrage. (If $\Pi_0^{manager}[X] < \Pi_0^{model}[X]$ a similar procedure can be used to create an arbitrage.)

4. (a) A short rate model is said to have an affine term structure if zero coupon bond prices can be written on the following form

$$p(t,T) = e^{A(t,T)-B(t,T)r_t},$$

where A and B are deterministic functions.

Sufficient conditions on μ and σ which guarantee the existence of an affine term structure are that μ and σ^2 are affine in r (and that there exists solutions to two certain ordinary differential equation, see below), i.e.

$$\begin{cases} \mu(t,r) &= a(t)r + b(t), \\ \sigma^2(t,r) &= c(t)r + d(t). \end{cases}$$

To see this insert these expression into the term structure equation

$$\begin{cases} F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - rF^T = 0, \\ F(T,r) = 1. \end{cases}$$

(Here $F^T(t,r_t) = p(t,T)$, and we have used the notation $F_t^T = \partial F^T/\partial t$, etc.) This will after some rewriting give you

$$\begin{cases} A_t(t,T) - b(t)B(t,T) + \frac{1}{2}d(t)B^2(t,T) \\ + \Big(1 + B_t(t,T) + a(t)B(t,T) - \frac{1}{2}c(t)B^2(t,T)\Big)\, r = 0, \\ e^{A(T,T)-B(T,T)r} = 0. \end{cases}$$

This equation should hold for all t and r, so there will be an affine term structure if A and B solve the following ordinary differential equations

$$\begin{cases} A_t(t,T) &= b(t)B(t,T) - \frac{1}{2}d(t)B^2(t,T), \\ A(T,T) &= 0, \end{cases} \tag{1}$$

and

$$\begin{cases} B_t(t,T) + a(t)B(t,T) - \dfrac{1}{2}c(t)B^2(t,T) &= -1, \\ B(T,T) &= 0. \end{cases} \tag{2}$$

(b) Use Itô's formula on $p(t,T) = e^{A(t,T)-B(t,T)r_t}$ to obtain

$$dp(t,T) = \dots dt - \sigma(t,r_t)B(t,T)p(t,T)dV_t.$$

The volatility is thus

$$\nu(t,T) = -\sigma(t,r_t)B(t,T),$$

where the function B is the solution to equation (2).

(c) It is readily verified that the Vasiček model satisfies the sufficient conditions from (a) and thus has an affine term structure, i.e. zero coupon bond prices are of the form

$$p(t,T) = e^{A(t,T)-B(t,T)r_t}. \tag{3}$$

Identifying parameters we see that A and B should solve the following equations

$$\begin{cases} A_t(t,T) &= bB(t,T) - \frac{1}{2}\sigma^2 B^2(t,T), \\ A(T,T) &= 0, \end{cases}$$

and

$$\begin{cases} B_t(t,T) - aB(t,T) &= -1, \\ B(T,T) &= 0. \end{cases}$$

Solving the equation for B yields

$$B(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right).$$

A can then be obtain via integration

$$A(t,T) = \int_t^T \frac{1}{2}\sigma^2 B^2(s,T)ds - \int_t^T bB(s,T)ds.$$

5. (a) The zero coupon bond with maturity T.

(b) Under Q^T every price process normalized by the price process of the zero coupon bond maturing at time T should be a martingale. In particular the process $Z^T(t) = B(t)/p(t,T)$ should be a martingale.

Using Itô's formula on $Z^T(t) = B(t)/p(t,T)$ we obtain the Q-dynamics

$$\begin{aligned} dZ_t^T &= \frac{1}{p_t^T}dB_t - \frac{B_t}{(p_t^T)^2}dp_t^T + \frac{1}{2}2\frac{B_t}{(p_t^T)^3}(dp_t^T)^2 \\ &= (\nu_t^T)^2 Z_t^T dt - \nu_t^T Z_t^T dV_t \end{aligned}$$

Now define a Girsanov transformation by

$$dQ^T = L^T(t)dQ, \qquad \text{on } \mathcal{F}_t,$$

where

$$\begin{cases} dL_t^T &= g_t^T L_t^T dV_t, \\ L_0^T &= 1. \end{cases}$$

From Girsanov's theorem we have that

$$dV_t = g_t^T dt + dU_t^T,$$

where U^T is a Q^T-Wiener process. We then have the Q^T-dynamics of Z^T as

$$\begin{aligned} dZ_t^T &= (\nu_y^T)^2 Z_t^T dt - \nu_t^T Z_t^T (g_t^T dt + dU_t^T) \\ &= \left((\nu_t^T)^2 - \nu_t^T g_t^T\right) Z_t^T dt - \nu_t^T Z_t^T dU_t^T \end{aligned}$$

In order for Z^T to be a Q^T-martingale the drift must be zero and this means that we must have

$$g^T(t) = \nu(t,T).$$

An explicit expression for the likelihood process L^T is therefore

$$L_t^T = \exp\left\{\int_0^t \nu(s,T)dV_s - \frac{1}{2}\int_0^t \nu^2(s,T)ds\right\}.$$

(c) From the martingale property of Π/p^T under Q^T (here Π denotes the price process of a traded asset) we have that

$$\frac{\Pi(t)}{p(t,T)} = E^T\left[\left.\frac{\Pi(T)}{p(T,T)}\right| \mathcal{F}_t\right].$$

Using that $P(T,T) = 1$ and that $\Pi(T) = X$ for a T-claim X we obtain the following pricing formula after some rewriting

$$\Pi(t) = p(t,T)E^T\left[X|\,\mathcal{F}_t\right].$$

(d) In order to be able to use the pricing formula from (c) we need the dynamics of r under Q^T. From (b) we know that the Girsanov kernel from Q to Q^T is given by the volatility of the zero coupon bond with maturity T, and from exercise 4 (b) we know that this is given by

$$\nu(t,T) = -\sigma B(t,T),$$

where we know from exercise 4 (c) that

$$B(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right). \tag{4}$$

This means that the Q^T-dynamics of r are given by

$$dr_t = \left(b - \sigma^2 B_t^T - ar_t\right)dt + \sigma dU_t^T.$$

An explicit solution of this SDE can be found in the hints and is given by

$$r_T = e^{-a(T-t)}r_t + \int_t^T e^{-a(T-s)}\left(b - \sigma^2 B_s^T\right)ds + \int_t^T e^{-a(T-s)}\sigma dU_s^T$$

From the above formula we see that the expectation of r_T conditional on $\mathcal{F}_t$ is

$$E^T[r_T|\mathcal{F}_t] = e^{-a(T-t)}r_t + \int_t^T e^{-a(T-s)}\left(b - \sigma^2 B_s^T\right)ds.$$

The price of the claim is thus given by

$$\Pi(t) = p(t,T)\left[e^{-a(T-t)}r_t + \int_t^T e^{-a(T-s)}\left(b - \sigma^2 B_s^T\right)ds\right],$$

where $p(t,T)$ is given by formula (3) computed in exercise 4 (c) and B^T is given by (4).