TENTAMEN I 5B1580 RISKVÄRDERING OCH RISKHANTERING FÖR F4 OCH I3 FREDAG DEN 18 MARS 2005 KL 14.00–19.00.

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Tillåtna hjälpmedel: Inga.

Införda beteckningar skall förklaras och definieras. Resonemang och uträkningar skall vara så utförliga att de är lätta att följa.

Resultatet anslås senast fredagen den 8 april 2005 på Matematisk statistiks anslagstavla i entréplanet, Lindstedtsvägen 25, rakt fram innanför porten.

Varje korrekt lösning ger tio poäng. Gränsen för godkänt är preliminärt 25 poäng.

Tentamen kommer att finnas tillgänglig på elevexpeditionen sju veckor efter skrivningstillfället.

Lycka till!

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Stock	day01	day02	day03	day04	day05	day06	day07	day08	day09	day10
A	0.001	-0.008	0.002	0.013	0.007	-0.019	0.019	-0.003	-0.002	-0.005
B	0.009	-0.006	-0.003	0.009	0.000	-0.034	0.021	0.002	-0.007	0.001

Stock	day11	day12	day13	day14	day15	day16	day17	day18	day19	day20
A	-0.012	-0.013	-0.002	0.013	0.001	0.000	-0.002	-0.013	-0.006	0.003
B	-0.010	-0.018	-0.005	0.007	0.000	0.000	-0.014	-0.020	-0.006	0.007

Table 1: Observed daily log-returns for stocks A and B.

Problem 1

(a) Figure 1 shows three plots A, B and C. Plot A shows a scatter plot of BMW and DAX log returns. Suppose you have a portfolio consisting of a long position of one share of BMW and a short position of one share of DAX, i.e. the portfolio value at time t is given by $V_t = 1 \cdot S_{\text{BMW},t} - 1 \cdot S_{\text{DAX},t}$. Which one of Plots B and C shows linearized losses of your portfolio? Motivate your answer properly! (5 p)

(b) Consider a portfolio consisting of a long position of 2 shares of stock A and a short position of 1 share of stock B. The stock prices today are given by $(S_A, S_B) = (100, 200)$. In Table 1 you find 20 bivariate observations $(x_{A,1}, x_{B,1}), \ldots, (x_{A,20}, x_{B,20})$ of daily log returns from random vectors $(X_{A,1}, X_{B,1}), \ldots, (X_{A,20}, X_{B,20})$ which are assumed to be iid. Use historical simulation to compute the empirical estimate of

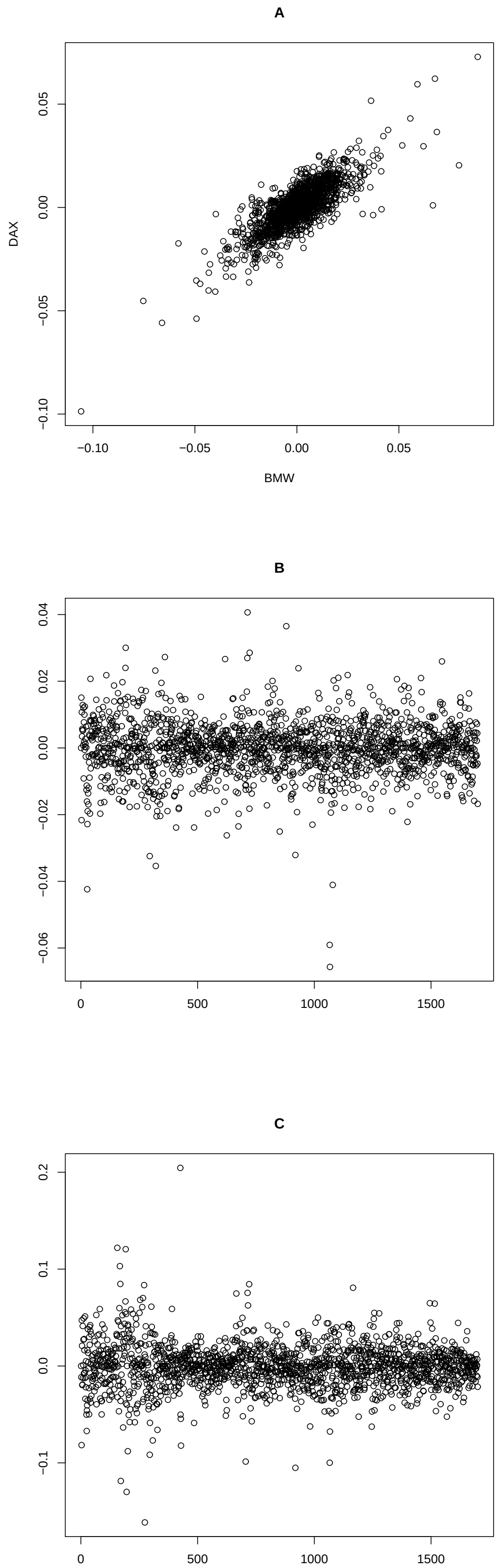

Figure 1: Plots A, B and C from upper to lower.

$\mathrm{VaR}_{0.95}(L^{\Delta})$ and $\mathrm{VaR}_{0.99}(L^{\Delta})$, where L^{Δ} denotes the linearized portfolio loss over the time period today-until-tomorrow. Recall that the empirical VaR estimator can be written as

$$\widehat{\mathrm{VaR}}_{p,\mathrm{E}}(Z) = Z_{[n(1-p)]+1,n}, \quad p \in (0,1),$$

where $Z_{1,n} > \cdots > Z_{n,n}$ and $[y] = \sup\{m \in \{0,1,2,\dots\} : m \le y\}$, i.e. the integer part of y. (5 p)

Problem 2

On average an insurance company has to pay 2 Million Swedish Crownes each year to meet the claims of its policy holders. Let L denote the (unknown) amount in Million Swedish Crownes that the insurance company will have to pay next year. Consider two models for L, namely,

$$\begin{array}{lll} (A) & \mathrm{P}(L \le x) = 1 - e^{-cx} & (c, x > 0), \\ (B) & \mathrm{P}(L \le x) = 1 - x^{-\alpha} & (\alpha, x > 1). \end{array}$$

Compute $\mathrm{VaR}_{0.99}(L)$ and $\mathrm{VaR}_{0.9999}(L)$ for the two models (A) and (B). You should give numeric results! (10 p)

You may use that: $\ln 5 \approx 1.61$, $\ln 10 \approx 2.30$, $\ln 50 \approx 3.91$, $\ln 100 \approx 4.61$, $\ln 500 \approx 6.21$, $\ln 1000 \approx 6.91$, $\ln 5000 \approx 8.52$, $\ln 10000 \approx 9.21$.

Problem 3

You have bought one share of three different stocks A, B and C with stock prices 100 today and unknown stock prices S_A, S_B and S_C in one year from today. In order to determine the riskiness of holding your portfolio one year an expert determines five possible scenarios and assigns probabilities to them according to:

$$\begin{array}{ll} (1) & \mathrm{P}((S_A, S_B, S_C) = (130, 110, 100)) = 0.2, \\ (2) & \mathrm{P}((S_A, S_B, S_C) = (110, 100, 110)) = 0.3, \\ (3) & \mathrm{P}((S_A, S_B, S_C) = (95, 90, 105)) = 0.43, \\ (4) & \mathrm{P}((S_A, S_B, S_C) = (90, 90, 100)) = 0.05, \\ (5) & \mathrm{P}((S_A, S_B, S_C) = (70, 90, 80)) = 0.02. \end{array}$$

(a) Compute the distribution of the portfolio loss L and draw its distribution function. (5 p)

(b) Compute $\mathrm{VaR}_{0.95}(L)$. (5 p)

Problem 4

Let X be a financial loss with distribution function F satisfying $\overline{F} = 1 - F \in \mathrm{RV}_{-\alpha}$ $(\alpha > 0)$, i.e. $\overline{F}$ is a regularly varying function.

(a) Compute $\lim_{u\to\infty} \mathrm{P}(X - u > ux/\alpha \mid X > u)$ for $x > 0$. (3 p)

(b) Find a sequence (a_n) such that $\lim_{n\to\infty} n\overline{F}(a_n x) = x^{-\alpha}$ for $x > 0$. You may assume that F is continuous and strictly increasing. (3 p)

(c) Let (X_k) be a sequence of iid random variables satisfying $X_k \stackrel{\mathrm{d}}{=} X$ (equal in distribution) for each k. Show that $\lim_{n\to\infty} n\overline{F}(a_n x) = x^{-\alpha}$ for $x > 0$ implies

$$\lim_{n\to\infty} \mathrm{P}(a_n^{-1} \max\{X_1, \dots, X_n\} \le x) = H(x), \quad x > 0,$$

and determine $H(x)$. (3 p)

(d) Hill estimation on an iid sample of size n from the distribution of X gives you the Hill estimate

$$\Big(\frac{1}{k}\sum_{j=1}^{k}(\ln x_{j,n} - \ln x_{k,n})\Big)^{-1} \approx 2,$$

where $x_{1,n} > \cdots > x_{n,n}$ and k is assumed to be optimally chosen. Use this information to approximate $\mathrm{P}(X > 2 \cdot 10^6 \mid X > 10^6)$. (1 p)

Problem 5

Let $\mathbf{X} = (X_1, X_2)^T$ be a vector of log returns, for the time period today-until-tomorrow, of two stocks with todays stock prices $S_1 = S_2 = 1$. Suppose that $\mathbf{X} \sim E_2(\boldsymbol{\mu}, \Sigma, \psi)$ (elliptically distributed) with linear correlation coefficient ρ and that

$$\boldsymbol{\mu} = \mathbf{0}, \quad \Sigma = \begin{pmatrix} \sigma^2 & \sigma^2\rho \\ \sigma^2\rho & \sigma^2 \end{pmatrix}.$$

Your total capital is 1 which you want to invest fully in the two stocks giving you a linearized portfolio loss $L^{\Delta} = L^{\Delta}(w_1, w_2)$ where w_1 and w_2 are portfolio weights. Two investment strategies are available (long positions):
(A) invest your money in equal shares in the two stocks: $w_{A1} = w_{A2} = 1/2$;
(B) invest all your money in the first stock: $w_{B1} = 1, w_{B2} = 0$.

Compute the ratio $\mathrm{VaR}_{0.99}(L_A^{\Delta})/\mathrm{VaR}_{0.99}(L_B^{\Delta})$, where L_A^{Δ} and L_B^{Δ} are linearized losses for investment strategies A and B, respectively. (10 p)

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Problem 1

(a) The portfolio loss L and the linearized loss L^{Δ} are given by

$$\begin{aligned} L &= -\Big(1\cdot[\exp\{X_{\text{BMW}}\}-1] - 1\cdot[\exp\{X_{\text{DAX}}\}-1]\Big),\\ L^{\Delta} &= -\Big(1\cdot X_{\text{BMW}} - 1\cdot X_{\text{DAX}}\Big) = -X_{\text{BMW}} + X_{\text{DAX}}. \end{aligned}$$

In plot A we see that X_{BMW} and X_{DAX} have a strong positive dependence, particularly in the upper right and lower left tail of the bivariate distribution. Since the portfolio consists of one long a one short position and the large losses/gains are of similar size it is likely that $X_{\text{BMW}} \approx X_{\text{DAX}}$ so that $L^{\Delta} = -X_{\text{BMW}} + X_{\text{DAX}}$ is small. Therefore, it must be Plot B that shows linearized losses from our portfolio.

(b) Portfolio losses using the historical simulation approach:

$$\begin{aligned} l_k &= -2\cdot 100\cdot(\exp(x_k)-1) + 1\cdot 200\cdot(\exp(y_k)-1),\\ l_k^{\Delta} &= -200x_k + 200y_k, \end{aligned}$$

for $k = 1,\dots,20$, where the (x_k, y_k)'s are shown in Table 1. Let $l^{\Delta}_{1,20} > l^{\Delta}_{2,20} > \dots > l^{\Delta}_{20,20}$ denote the ordered linearized portfolio losses. The empirical VaR estimate for the linearized loss is

$$\widehat{\text{VaR}}_{p,\text{E}} = l^{\Delta}_{[20(1-p)]+1,20}.$$

By inspecting Table 1 we find that

$$\begin{aligned} l^{\Delta}_{1,20} &= -200\cdot 0.001 + 200\cdot 0.009 = 1.6 \quad \text{(Day 1)},\\ l^{\Delta}_{2,20} &= -200\cdot(-0.005) + 200\cdot 0.001 = 1.2. \quad \text{(Day 10)}. \end{aligned}$$

Note that $[20(1-0.95)]+1 = 2$ and $[20(1-0.99)]+1 = 1$. Hence,

$$\widehat{\text{VaR}}_{0.95} = l^{\Delta}_{2,20} = 1.2 \quad \text{and} \quad \widehat{\text{VaR}}_{0.99} = l^{\Delta}_{1,20} = 1.6.$$

Problem 2

We have $E(L) = 2$. We must find c and α so that $E(L_A) = E(L_B) = 2$. We have

$$E(L_A) = 1/c, \quad E(L_B) = \alpha/(\alpha-1).$$

Hence, $c = 1/2$ and $\alpha = 2$. Moreover,

$$\begin{aligned} F^{-1}_{L_A}(u) &= -\frac{1}{c}\ln(1-u) = -2\ln(1-u),\\ F^{-1}_{L_B}(u) &= (1-u)^{-1/\alpha} = (1-u)^{-1/2}. \end{aligned}$$

Since $\mathrm{VaR}_u(L_A) = F_{L_A}^{-1}(u)$ and $\mathrm{VaR}_u(L_B) = F_{L_B}^{-1}(u)$ we have

$$\begin{aligned}
\mathrm{VaR}_{0.99}(L_A) &= -2\ln(1/100) = 2\ln(100) \approx 9.22, \\
\mathrm{VaR}_{0.9999}(L_A) &= -2\ln(1/10000) = 2\ln(10000) \approx 18.42, \\
\mathrm{VaR}_{0.99}(L_B) &= \sqrt{100} = 10, \\
\mathrm{VaR}_{0.9999}(L_B) &= \sqrt{10000} = 100.
\end{aligned}$$

Problem 3

We have $L = -(S_A - 100) - (S_B - 100) - (S_C - 100)$. Hence,

$$L = \begin{cases} -40 & \text{with probability } 0.2, \\ -20 & \text{with probability } 0.3, \\ +10 & \text{with probability } 0.43, \\ +20 & \text{with probability } 0.05, \\ +60 & \text{with probability } 0.02. \end{cases}$$

The distribution function F is a right-continuous step function satisfying:

$$F(x) = \begin{cases} 0 & x \in (-\infty, -40), \\ 0.2 & x \in [-40, -20), \\ 0.5 & x \in [-20, 10), \\ 0.93 & x \in [10, 20), \\ 0.98 & x \in [20, 60), \\ 1 & x \in [60, \infty). \end{cases}$$

(b) We have

$$\mathrm{VaR}_p(L) = \inf\{x \in \mathbb{R} : F(x) \geq p\} = \min\{x \in \mathbb{R} : F(x) \geq p\}.$$

Hence, $\mathrm{VaR}_{0.95}(L) = \min\{x \in \mathbb{R} : F(x) \geq 0.95\} = 20$.

Problem 4

(a) We have

$$\begin{aligned}
\mathrm{P}(X - u > ux/\alpha \mid X > u) &= \frac{\mathrm{P}(X > u(1 + x/\alpha))}{\mathrm{P}(X > u)} \\
&= \frac{\overline{F}(u(1 + x/\alpha))}{\overline{F}(u)} \\
&\to (1 + x/\alpha)^{-\alpha},
\end{aligned}$$

using that $\overline{F}$ is regularly varying with index α.

(b) $\lim_{n\to\infty} n\overline{F}(a_n) = \lim_{n\to\infty} n(1 - F(a_n)) = 1$. Hence one choice is a_n such that $F(a_n) = 1 - 1/n$, i.e. we can choose $a_n = F^{-1}(1 - 1/n)$. Now,

$$n\overline{F}(a_n x) = \frac{\overline{F}(a_n x)}{1/n} = \frac{\overline{F}(a_n x)}{1 - F(F^{-1}(1 - 1/n))} = \frac{\overline{F}(a_n x)}{\overline{F}(a_n)} \to x^{-\alpha}.$$

(c)

$$\begin{aligned}\mathrm{P}(\max\{X_1,\dots,X_n\} \le a_n x) &= F^n(a_n x) = (1-\overline{F}(a_n x))^n \\ &= \Big(1 - \frac{n\overline{F}(a_n x)}{n}\Big)^n \to \exp\{-x^{-\alpha}\} \\ &= H(x).\end{aligned}$$

(d) From the Hill plot we find that $\alpha \approx 2$. Hence,

$$\mathrm{P}(X > 2\cdot 10^6 \mid X > 10^6) \approx 2^{-\alpha} \approx 1/4.$$

Problem 5

We have

$$L^{\Delta} = -\mathbf{w}^T\mathbf{X} \stackrel{\mathrm{d}}{=} \mathbf{w}^T\mathbf{X}$$

since $\mathbf{X} \sim E_2(\mathbf{0}, \Sigma, \psi)$. Moreover, $\mathbf{w}^T\mathbf{X} \sim E_1(0, \mathbf{w}^T\Sigma\mathbf{w}, \psi)$. Hence, $\mathbf{w}^T\mathbf{X} \stackrel{\mathrm{d}}{=} \sqrt{\mathbf{w}^T\Sigma\mathbf{w}}Z$, where $Z \sim E_1(0,1,\psi)$. Hence, $\mathrm{VaR}_p(\mathbf{w}^T\mathbf{X}) = \sqrt{\mathbf{w}^T\Sigma\mathbf{w}}\,\mathrm{VaR}_p(Z)$. This yields

$$\frac{\mathrm{VaR}_{0.99}(\mathbf{w}_A^T\mathbf{X})}{\mathrm{VaR}_{0.99}(\mathbf{w}_B^T\mathbf{X})} = \frac{\sqrt{\mathbf{w}_A^T\Sigma\mathbf{w}_A}}{\sqrt{\mathbf{w}_B^T\Sigma\mathbf{w}_B}}.$$

We have $\mathbf{w}_A^T\Sigma\mathbf{w}_A = \sigma^2(1+\rho)/2$ and $\mathbf{w}_B^T\Sigma\mathbf{w}_B = \sigma^2$. Hence

$$\frac{\mathrm{VaR}_{0.99}(L_A^{\Delta})}{\mathrm{VaR}_{0.99}(L_B^{\Delta})} = \frac{\mathrm{VaR}_{0.99}(\mathbf{w}_A^T\mathbf{X})}{\mathrm{VaR}_{0.99}(\mathbf{w}_B^T\mathbf{X})} = \sqrt{(1+\rho)/2}.$$