TENTAMEN I 5B1580 RISKVÄRDERING OCH RISKHANTERING FÖR F4 OCH I3 MÅNDAGEN DEN 19 DECEMBER 2005 KL 14.00–19.00.

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Tillåtna hjälpmedel: Inga.

Införda beteckningar skall förklaras och definieras. Resonemang och uträkningar skall vara så utförliga att de är lätta att följa.

Resultatet anslås senast fredagen den 13 januari 2006 på Matematisk statistiks anslagstavla i entréplanet, Lindstedtsvägen 25, rakt fram innanför porten.

Varje korrekt lösning ger tio poäng. Gränsen för godkänt är preliminärt 25 poäng.

Tentamen kommer att finnas tillgänglig på elevexpeditionen sju veckor efter skrivningstillfället.

Lycka till!

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Problem 1

You hold a portfolio consisting of a long position of 1 share of stock A. The stock price today is $S_0 = 100$ SEK. The daily log returns $X_1, X_2, \dots$ of stock A are assumed to be independent and normally distributed with zero mean and standard deviation $\sigma = 0.1$. With Φ denoting the distribution function of $N(0,1)$ (standard normal distribution) it holds that $\Phi^{-1}(0.99) \approx 2.33$.

(a) Let L_1 be the 1-day portfolio loss. Compute $\text{VaR}_{0.99}(L_1)$ (Value-at-Risk). (5 p)

(b) Let L_{10} be the 10-day portfolio loss. Compute $\text{VaR}_{0.99}(L_{10})$ (Value-at-Risk). (5 p)

In (a) and (b) use some of the following information:
$e^{-2.33} \approx 0.097$, $e^{-0.233} \approx 0.79$, $e^{-0.0233} \approx 0.98$, $e^{-2.33/\sqrt{10}} \approx 0.48$, $e^{-0.233/\sqrt{10}} \approx 0.93$.

Problem 2

Consider an insurance loss L with distribution function $\text{P}(L \le l) = 1 - l^{-2}$ for $l \ge 1$. Compute $\text{ES}_{0.99}(L)$ (Expected Shortfall). (10 p)

Problem 3

Consider a portfolio loss L with distribution function F. It is assumed that $\overline{F} = 1 - F \in \mathrm{RV}_{-2}$, i.e. $\overline{F}$ is regularly varying at infinity with index -2. To account for stochastic volatility you introduce the random variable σ with distribution

$$\mathrm{P}(\sigma = 1/2) = 3/4, \quad \mathrm{P}(\sigma = 2) = 1/4,$$

which represents two possible volatility regimes. It is assumed that σ and L are independent.
Which of the models L and σL gives asymptotically the biggest probability for large losses? i.e. compute the limit $\lim_{z\to\infty} \mathrm{P}(\sigma L > z)/\mathrm{P}(L > z)$. (10 p)

Problem 4

Consider a homogeneous portfolio with 100 loans and let N be the total number of defaults one year from now. To model the default risk we consider the CreditRisk+ model with one single Gamma(α, β)-distributed risk factor Z with density function

$$f_Z(z) = \frac{z^{\alpha-1}e^{-z/\beta}}{\beta^\alpha \Gamma(\alpha)}, \quad z > 0, \alpha > 0, \beta > 0,$$

and mean $E(Z) = \alpha\beta$. We assume that the CreditRisk+ model is chosen so that it is a Poisson mixture model with $\lambda_i(z) = z/100$ for $i = 1, \dots, 100$. Moreover, $\alpha = 1/\beta$. Hence, conditional on Z, the default indicators are independent and Poisson$(Z/100)$-distributed.

(a) Compute the expected number of defaults, i.e. $E(N)$. (5 p)

(b) Compute the probability the there will be no defaults, i.e. $\mathrm{P}(N = 0)$. (5 p)

Recall: Y is Poisson(μ)-distributed if $\mathrm{P}(Y = k) = \mu^k e^{-\mu}/k!$ for $k \geq 0$.

Problem 5

A bank has a loan portfolio of n loans and wants to analyze its 1-year portfolio default risk. Loan k defaults if the asset value Y_k of obligor k falls below its debt d_k, i.e. if $Y_k \leq d_k$. Every loan has the same default probability $p \in (0, 1)$. The asset value Y_k is Lognormally distributed with

$$Y_k \overset{\mathrm{d}}{=} e^Z \quad \text{(equally distributed)},$$

where Z is normally distributed with zero mean and variance $1/4$. The vector $(Y_1, \dots, Y_n)$ has a Clayton copula given by

$$C^{\mathrm{Cl}}(u_1, \dots, u_n) = (u_1^{-1} + \dots + u_n^{-1} - n + 1)^{-1}.$$

Compute $\lim_{p\downarrow 0} \mathrm{P}(Y_k \leq d_k \text{ for } k = 1, \dots, n \mid Y_1 \leq d_1)$. (10 p)

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Problem 1

(a) Notice that

$$L_1 = -100(e^{X_1} - 1) \stackrel{\mathrm{d}}{=} -100(e^{\sigma Z} - 1),$$

where $Z \sim N(0,1)$. We have $\mathrm{VaR}_u(L_1) = F_{L_1}^{-1}(u)$ and $F_{L_1}(F_{L_1}^{-1}(u)) = u$. Moreover,

$$\begin{aligned} F_{L_1}(l) &= \mathrm{P}(-100(e^{\sigma Z} - 1) \leq l) \\ &= \mathrm{P}(e^{\sigma Z} \geq 1 - l/100) \\ &= \mathrm{P}(Z \geq \sigma^{-1} \ln(1 - l/100)) \\ &= 1 - \Phi(\sigma^{-1} \ln(1 - l/100)). \end{aligned}$$

Hence,

$$\begin{aligned} & 1 - \Phi(\sigma^{-1} \ln(1 - F_{L_1}^{-1}(u)/100)) = u \\ \Leftrightarrow \quad & \ln(1 - F_{L_1}^{-1}(u)/100) = \sigma \Phi^{-1}(1-u) \\ \Leftrightarrow \quad & 1 - F_{L_1}^{-1}(u)/100 = e^{\sigma \Phi^{-1}(1-u)} \\ \Leftrightarrow \quad & F_{L_1}^{-1}(u) = 100(1 - e^{\sigma \Phi^{-1}(1-u)}). \end{aligned}$$

Since Z is symmetric about 0 we have $\Phi^{-1}(1-u) = -\Phi^{-1}(u)$. Hence,

$$\begin{aligned} \mathrm{VaR}_{0.99}(L_1) &= 100(1 - e^{-\sigma \Phi^{-1}(0.99)}) \\ &\approx 100(1 - e^{-0.233}) \approx 21. \end{aligned}$$

(b) Notice that

$$L_{10} = -100(e^{X_1 + \cdots + X_{10}} - 1) \stackrel{\mathrm{d}}{=} -100(e^{\sqrt{10} X_1} - 1) \stackrel{\mathrm{d}}{=} -100(e^{Z/\sqrt{10}} - 1),$$

where $Z \sim N(0,1)$. We have $\mathrm{VaR}_u(L_{10}) = F_{L_{10}}^{-1}(u)$. With computations similar to those in (a) we get

$$\mathrm{VaR}_{0.99}(L_{10}) = 100(1 - e^{-\Phi^{-1}(0.99)/\sqrt{10}}) \approx 100(1 - e^{-2.33/\sqrt{10}}) \approx 52.$$

Problem 2

Denote by F and f the distribution function and density of L. We have $F(l) = 1 - l^{-2}$ and hence $f(l) = 2l^{-3}$ and $F^{-1}(u) = \mathrm{VaR}_u(L) = (1-u)^{-1/2}$. Hence, $\mathrm{VaR}_{0.99}(L) = 10$ and

$$\begin{aligned} \mathrm{ES}_{0.99}(L) &= \frac{1}{1 - 0.99} \int_{\mathrm{VaR}_{0.99}(L)}^{\infty} l f(l) \mathrm{d}l \\ &= 100 \int_{10}^{\infty} 2l^{-2} \mathrm{d}l = 200/10 = 20. \end{aligned}$$

Problem 3

$$\begin{aligned}\lim_{z\to\infty}\frac{\mathrm{P}(\sigma L>z)}{\mathrm{P}(L>z)} &= \lim_{z\to\infty}\frac{3}{4}\frac{\mathrm{P}(L>2z)}{\mathrm{P}(L>z)}+\lim_{z\to\infty}\frac{1}{4}\frac{\mathrm{P}(L>z/2)}{\mathrm{P}(L>z)}\\ &= \frac{3}{4}2^{-2}+\frac{1}{4}2^{2}=\frac{3}{16}+1>1.\end{aligned}$$

Hence, the model σL gives loss probabilities that are bigger than those for the model L (asymptotically).

Problem 4

(a)

$$\begin{aligned}E(N) &= E(E(N\mid Z)) = 100E(E(X_1\mid Z))\\ &= 100E(0.01Z)=E(Z)=(1/\beta)\beta=1.\end{aligned}$$

(b)

$$\begin{aligned}\mathrm{P}(N=0) &= E(\mathrm{P}(N=0\mid Z)) = \int_0^\infty \mathrm{P}(N=0\mid Z=z)f_Z(z)\mathrm{d}z\\ &= \int_0^\infty (e^{-0.01z})^{100}\frac{z^{\alpha-1}e^{-z/\beta}}{\beta^\alpha\Gamma(\alpha)}\mathrm{d}z\\ &= \int_0^\infty \frac{z^{\alpha-1}e^{-z(1+1/\beta)}}{\beta^\alpha\Gamma(\alpha)}\mathrm{d}z\\ &= \int_0^\infty \frac{z^{\alpha-1}e^{-z/\widetilde{\beta}}}{\beta^\alpha\Gamma(\alpha)}\mathrm{d}z\\ &= \frac{\widetilde{\beta}^\alpha}{\beta^\alpha}\int_0^\infty \frac{z^{\alpha-1}e^{-z/\widetilde{\beta}}}{\widetilde{\beta}^\alpha\Gamma(\alpha)}\mathrm{d}z\\ &= \frac{\widetilde{\beta}^\alpha}{\beta^\alpha},\end{aligned}$$

where $\widetilde{\beta}=\beta/(1+\beta)$. Hence,

$$\mathrm{P}(N=0)=\frac{\widetilde{\beta}^\alpha}{\beta^\alpha}=(1+\beta)^{-\alpha}=(1+\beta)^{-1/\beta}.$$

Problem 5

Set $F_k(x) = \mathrm{P}(Y_k \le x)$. We have

$$\begin{aligned}
&\lim_{p \downarrow 0} \mathrm{P}(Y_k \le d_k \text{ for } k = 1, \dots, n \mid Y_1 \le d_1) \\
&\quad = \lim_{p \downarrow 0} \frac{\mathrm{P}(Y_k \le d_k \text{ for } k = 1, \dots, n)}{\mathrm{P}(Y_1 \le d_1)} \\
&\quad = \lim_{p \downarrow 0} \frac{\mathrm{P}(F_k(Y_k) \le F_k(d_k) \text{ for } k = 1, \dots, n)}{\mathrm{P}(F_1(Y_1) \le F_1(d_1))} \\
&\quad = \lim_{p \downarrow 0} \frac{\mathrm{P}(F_k(Y_k) \le p \text{ for } k = 1, \dots, n)}{\mathrm{P}(F_1(Y_1) \le p)} \\
&\quad = \lim_{p \downarrow 0} \frac{C^{\mathrm{Cl}}(p, \dots, p)}{p} \\
&\quad = \lim_{p \downarrow 0} \frac{(n(1-p)/p + 1)^{-1}}{p} \\
&\quad = \lim_{p \downarrow 0} \frac{p}{n(1-p)p} = \frac{1}{n}.
\end{aligned}$$