TENTAMEN I 5B1580 RISKVÄRDERING OCH RISKHANTERING FÖR F4 OCH I3 MÅNDAGEN DEN 22 AUGUSTI 2005 KL 14.00–19.00.

Examinator: Filip Lindskog, tel. 790 7217, e-post: lindskog@math.kth.se

Tillåtna hjälpmedel: Inga.

Införda beteckningar skall förklaras och definieras. Resonemang och uträkningar skall vara så utförliga att de är lätta att följa.

Resultatet anslås senast fredagen den 9 september 2005 på Matematisk statistiks anslagstavla i entréplanet, Lindstedtsvägen 25, rakt fram innanför porten.

Varje korrekt lösning ger tio poäng. Gränsen för godkänt är preliminärt 25 poäng.

Tentamen kommer att finnas tillgänglig på elevexpeditionen sju veckor efter skrivningstillfället.

Lycka till!

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Problem 1

You hold a portfolio consisting of a long position of $\alpha = 5$ shares of stock A. The stock price today is $S_0 = 100$ SEK. The daily log return X of stock A from today until tomorrow is assumed to be normally distributed with zero mean and standard deviation $\sigma = 0.1$. Let L be the portfolio loss from today until tomorrow.

(a) What values can L take? (1 p)

(b) Find the function f such that $L = f(X)$. (3 p)

(c) Compute $\mathrm{VaR}_{0.99}(L)$. (6 p)

You may use that $F_X^{-1}(0.99) \approx 0.23$ and the information in Figure 1.

Problem 2

You hold the same portfolio as in Problem 1. You decide to keep your portfolio for 100 (trading) days before deciding what to do with the portfolio. The daily log returns are assumed to be independent and normally distributed with mean zero and standard deviation $\sigma = 0.1$.

(a) Compute $\mathrm{VaR}_{0.99}(L_{100})$, where L_{100} denotes the loss from today until 100 days

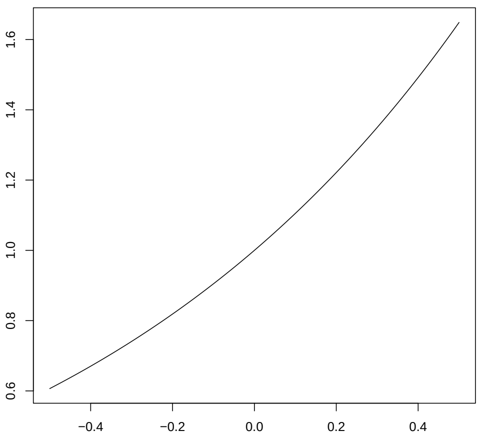

Figure 1: Plot of the function e^x for $x \in [-0.5, 0.5]$.

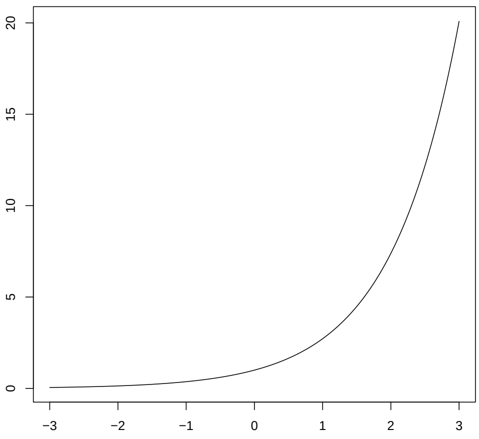

Figure 2: Plot of the function e^x for $x \in [-3, 3]$.

from today. (6 p)

(b) Compute $\mathrm{VaR}_{0.99}(L^{\Delta}_{100})$, where L^{Δ}_{100} denotes the corresponding linearized 100-day loss. (4 p)

You may use that $F_Z^{-1}(0.99) \approx 2.3$, where Z is normally distributed with mean zero and standard deviation one, and the information in Figure 2.

Problem 3

The random vector $\mathbf{X} = (X_1, X_2)^{\mathrm{T}}$ has an elliptical distribution with mean $E(\mathbf{X}) = \boldsymbol{\mu}$ and covariance matrix $\mathrm{Cov}(\mathbf{X}) = \Sigma$. It holds that

$$E(B(\mathbf{X} - \mathbf{b})) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathrm{Cov}(B(\mathbf{X} - \mathbf{b})) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where

$$\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}.$$

Compute the mean and covariance matrix of $\mathbf{X}$, i.e. determine $\boldsymbol{\mu}$ and Σ. (10 p)

Problem 4

Let X and Y represent losses in two lines of business (losses due to fire and car accidents) of an insurance company. Suppose that X and Y are independent. Suppose that X has the Pareto distribution $\mathrm{P}(X \le x) = 1 - x^{-\alpha}$ for $x > 1$ and $\alpha > 0$, and that Y is a positive random variable with $E(Y^k) < \infty$ for every $k > 0$.

Compute $\lim_{x\to\infty} \mathrm{P}(X > x \mid X + Y > x)$ and interpret the result. (10 p)

Hint: You may use Markov's inequality: $\mathrm{P}(Y > y) \le E(Y^{2\alpha})/y^{2\alpha}$ for every $y > 0$.

Problem 5

A bank has a loan portfolio of 100 loans. Let X_k be the default indicator for loan k such that $X_k = 1$ in case of default and 0 otherwise. The total number of defaults is $N = X_1 + \cdots + X_{100}$.

(a) Suppose that $X_1, \ldots, X_{100}$ are independent and identically distributed with $\mathrm{P}(X_1 = 1) = 0.01$. Compute $E(N)$ and $\mathrm{P}(N = n)$ for $n \in \{0, \ldots, 100\}$. (2 p)

(b) Consider the risk factor Z which reflects the state of the economy. Suppose that conditional on Z, the default indicators are independent and identically distributed with $\mathrm{P}(X_1 = 1 \mid Z) = Z$, where

$$\mathrm{P}(Z = 0.01) = 0.9 \quad \text{and} \quad \mathrm{P}(Z = 0.11) = 0.1.$$

Compute $E(N)$. (4 p)

(c) Consider the risk factor Z which reflects the state of the economy. Suppose that conditional on Z, the default indicators are independent and identically distributed with

$$\mathrm{P}(X_1 = 1 \mid Z) = Z^9,$$

where Z is uniformly distributed on $(0, 1)$. Compute $E(N)$. (4 p)

LÖSNING TILL TENTAMEN I 5B1580 RISKVÄRDERING OCH RISKHANTERING FÖR F4 OCH I3 2005-08-22

Problem 1

(a) $L \in (-\infty, 500]$ with probability 1.

(b) $L = -\alpha S_0(e^X - 1) = -500(e^X - 1)$ (see lecture notes). Hence, $f(x) = -500(e^x - 1)$.

(c) We have $\mathrm{VaR}_u(L) = F_L^{-1}(u)$ and $F_L(F_L^{-1}(u)) = u$.

$$\begin{aligned} F_L(l) &= \mathrm{P}(-500(e^X - 1) \le l) \\ &= \mathrm{P}(e^X \ge 1 - l/500) \\ &= \mathrm{P}(X \ge \ln(1 - l/500)) \\ &= 1 - F_X(\ln(1 - l/500)). \end{aligned}$$

Hence,

$$\begin{aligned} & 1 - F_X(\ln(1 - F_L^{-1}(u)/500)) = u \\ \Leftrightarrow \quad & \ln(1 - F_L^{-1}(u)/500) = F_X^{-1}(1-u) \\ \Leftrightarrow \quad & 1 - F_L^{-1}(u)/500 = e^{F_X^{-1}(1-u)} \\ \Leftrightarrow \quad & F_L^{-1}(u) = 500(1 - e^{F_X^{-1}(1-u)}). \end{aligned}$$

Since X is symmetric about 0 we have $F_X^{-1}(1-u) = -F_X^{-1}(u)$. Hence, $F_L^{-1}(u) = 500(1 - e^{-F_X^{-1}(u)})$. Using that $F_X^{-1}(0.99) \approx 0.23$ and with the help of Figure 1,

$$\begin{aligned} F_L^{-1}(0.99) &= 500(1 - e^{-F_X^{-1}(0.99)}) \\ &\approx 500(1 - e^{-0.23}) \approx 500(1 - 0.8) = 100. \end{aligned}$$

Hence, $\mathrm{VaR}_{0.99}(L) \approx 100$ SEK.

Problem 2

We have

$$L_{100} = -\alpha S_0(e^{X_{100} - 1}),$$

where X_{100} is the 100-day log return. Notice that

$$\begin{aligned} X_{100} &= \ln S_{100}/S_0 = \ln S_{100} - \ln S_0 \\ &= \ln S_1/S_0 + \cdots + \ln S_{100}/S_{99}, \end{aligned}$$

i.e. X_{100} is a sum of 100 independent normally distributed random variables with mean zero and standard deviation 0.1. Hence, $X_{100} \stackrel{\mathrm{d}}{=} Z$, where Z is normally distributed with zero mean and standard deviation one.

(a) As in Problem 1 we have

$$\begin{aligned}\mathrm{VaR}_{0.99}(L_{100}) &= 500(1 - e^{-F_Z^{-1}(0.99)}) \\ &\approx 500(1 - e^{-2.3}).\end{aligned}$$

Using Figure 2 we find that $e^{-2.3} = 1/e^{2.3} \approx 0.1$. Hence, $\mathrm{VaR}_{0.99}(L_{100}) \approx 500(1 - e^{-2.3}) \approx 450$.

(b) We have $L_{100}^{\Delta} = -500Z$. Hence, $\mathrm{VaR}_{0.99}(L_{100}^{\Delta}) = 500F_Z^{-1}(0.99) \approx 500 \cdot 2.3 = 1150$. One sees that using the linearized loss here gives a very bad risk estimate.

Problem 3

Set $\mathbf{Y} = B(\mathbf{X} - \mathbf{b})$. Standard matrix inversion yields

$$B^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

Since $\mathbf{X} = \mathbf{b} + B^{-1}\mathbf{Y}$, we find that

$$\begin{aligned}E(\mathbf{X}) &= \mathbf{b} + B^{-1}E(\mathbf{Y}) = \mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \\ \mathrm{Cov}(\mathbf{X}) &= \mathrm{Cov}(B^{-1}\mathbf{Y}) = B^{-1}\mathrm{Cov}(\mathbf{Y})(B^{-1})^{\mathrm{T}} = B^{-1}(B^{-1})^{\mathrm{T}} \\ &= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}.\end{aligned}$$

Problem 4

We have, for every $\varepsilon \in (0, 1)$ and $x > 0$,

$$\begin{aligned}\mathrm{P}(X + Y > x) &= \mathrm{P}(X + Y > x, X > (1-\varepsilon)x) + \mathrm{P}(X + Y > x, X \le (1-\varepsilon)x) \\ &\le \mathrm{P}(X + Y > x, X > (1-\varepsilon)x) + \mathrm{P}(X + Y > x, Y > \varepsilon x) \\ &\le \mathrm{P}(X > (1-\varepsilon)x) + \mathrm{P}(Y > \varepsilon x).\end{aligned}$$

Hence,

$$\begin{aligned}1 &\le \frac{\mathrm{P}(X + Y > x)}{\mathrm{P}(X > x)} \\ &\le \frac{\mathrm{P}(X > (1-\varepsilon)x)}{\mathrm{P}(X > x)} + \frac{\mathrm{P}(Y > \varepsilon x)}{\mathrm{P}(X > x)} \\ &\le \frac{\mathrm{P}(X > (1-\varepsilon)x)}{\mathrm{P}(X > x)} + \frac{E(Y^{2\alpha})}{(\varepsilon x)^{2\alpha}\,\mathrm{P}(X > x)} \\ &\to (1-\varepsilon)^{-\alpha} + 0\end{aligned}$$

as $x \to \infty$. Since this is true for every $\varepsilon \in (0, 1)$, choosing ε arbitrarily small gives

$$\lim_{x\to\infty} \frac{\mathrm{P}(X + Y > x)}{\mathrm{P}(X > x)} = 1.$$

Hence,

$$\lim_{x\to\infty} \mathrm{P}(X > x \mid X+Y > x) = \lim_{x\to\infty} \frac{\mathrm{P}(X > x, X+Y > x)}{\mathrm{P}(X+Y > x)}$$
$$= \lim_{x\to\infty} \frac{\mathrm{P}(X > x)}{\mathrm{P}(X+Y > x)} = 1.$$

If the insurance company suffers a large loss, it is likely that this is due to a large loss in the fire insurance line only.

Problem 5

(a) We have $N \sim \text{Binomial}(100, 0.01)$. Hence, $E(N) = 100 \cdot 0.01 = 1$ and

$$\mathrm{P}(N = n) = \binom{100}{n} 0.01^n 0.99^{100-n}.$$

(b) We have $N \mid Z \sim \text{Binomial}(100, Z)$. Hence,

$$E(N) = E(E(N \mid Z)) = E(100Z) = 100E(Z)$$
$$= 100(0.01 \cdot 0.9 + 0.11 \cdot 0.1) = 0.9 + 1.1 = 2.$$

(c) We have $N \mid Z \sim \text{Binomial}(100, Z)$. Hence,

$$E(N) = E(E(N \mid Z)) = E(100Z^9) = 100E(Z^9)$$
$$= 100 \cdot 0.1 = 10.$$