TENTAMEN I 5B1580 RISKVÄRDERING OCH RISKHANTERING TORSDAGEN DEN 8 JUNI 2006 KL 14.00–19.00.

Examinator: Filip Lindskog, tel. 790 7217, e-post: lindskog@math.kth.se

Tillåtna hjälpmedel: Inga.

Införda beteckningar skall förklaras och definieras. Resonemang och uträkningar skall vara så utförliga att de är lätta att följa.

Resultatet anslås senast torsdagen den 29 juni 2006 på Matematisk statistiks anslagstavla i entréplanet, Lindstedtsvägen 25, rakt fram innanför porten.

Varje korrekt lösning ger tio poäng. Gränsen för godkänt är preliminärt 25 poäng.

Tentamen kommer att finnas tillgänglig på elevexpeditionen sju veckor efter skrivningstillfället.

Lycka till!

- - - - - - - - - - - - - - - - - - - -

Problem 1

Let X be a daily logreturn for the DAX stock price index. Empirical studies suggest that X has zero mean and standard deviation $1/100$. Moreover, it is assumed that X is symmetric around its mean so that X and $-X$ have the same distribution. Let $\mathrm{VaR}_{0.99}(X)$ be Value-at-Risk at the level 0.99 for the logreturn X. A trader claims that $\mathrm{VaR}_{0.99}(X) = 1/10$. Use Chebyshev's inequality

$$\mathrm{P}(|X - E[X]| > x) \leq E[(X - E[X])^2]/x^2, \quad x > 0,$$

to show that this statement is wrong. (10 p)

Problem 2

Consider a stock market with n stocks and corresponding daily logreturns $X_1, \dots, X_n$ from today until tomorrow. The number of stocks n **is assumed to be very large**. Assume that the logreturns are equally distributed and each with zero mean. Assume also that the linear correlation coefficient for each pair of (different) logreturns is $\rho > 0$ and that the joint distribution of $X_1, \dots, X_n$ is a multivariate t-distribution with 5 degrees of freedom. Consider two portfolios:
(A) Long position of $1/2$ shares in Stock 1, short position of $1/2$ shares in Stock 2.
(B) Long positions of $1/n$ shares in each stock.
For which values of ρ is it true that $\mathrm{VaR}_{0.99}(L_A^{\Delta}) > \mathrm{VaR}_{0.99}(L_B^{\Delta})$? $L_A^{\Delta}, L_B^{\Delta}$ denotes the linearized portfolio losses. (10 p)

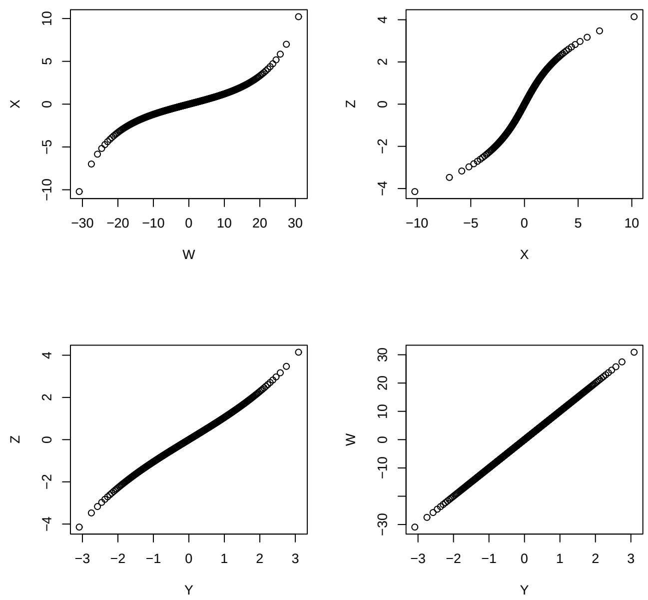

Figure 1: QQ-plots

Problem 3

Consider four iid samples of logreturns, each of size 500, for four different stocks. The samples $X = \{x_1, \ldots, x_{500}\}$, $Y = \{y_1, \ldots, y_{500}\}$, $Z = \{z_1, \ldots, z_{500}\}$ and $W = \{w_1, \ldots, w_{500}\}$ are compared in the QQ-plot in Figure 1. The logreturn distributions are: $N(0,1)$, $N(0,\sigma^2)$ with $\sigma > 1$, $t(3)$ and $t(10)$, where $t(\nu)$ denotes a standard t-distribution with ν degrees of freedom. Determine the correct match between the four samples and the four distributions. (10 p)

Problem 4

Consider a homogeneous portfolio with 100 loans and let N be the total number of defaults one year from now. To model the default risk we consider the CreditRisk+ model with one single Gamma(α,β)-distributed risk factor Z with density function

$$f_Z(z) = \frac{z^{\alpha-1}e^{-z/\beta}}{\beta^\alpha \Gamma(\alpha)}, \quad z > 0, \alpha > 0, \beta > 0,$$

and mean $E(Z) = \alpha\beta$. We assume that the CreditRisk+ model is chosen so that it is a Poisson mixture model with $\lambda_i(z) = z/100$ for $i = 1, \ldots, 100$. Moreover, $\alpha = 1/\beta$. Hence, conditional on Z, the default indicators are independent and Poisson$(Z/100)$-distributed.

(a) Compute the probability $\mathrm{P}(N = 1 \mid Z = 1)$. (4 p)

(b) Compute the probability the there will be exactly one default, i.e. compute $\mathrm{P}(N = 1)$. (6 p)

Hints: (1) A random variable Y is Poisson(μ)-distributed if $\mathrm{P}(Y = k) = \mu^k e^{-\mu}/k!$ for $k \geq 0$. (2) The Gamma function satisfies $\Gamma(z+1) = z\Gamma(z)$ for $z > 0$.

Problem 5

French and German companies have bought fire insurance from a insurance company in Riskyland with local currency Risky. Let U and V be potential claim sizes in (million) Euro from French and German fire claims next year. Let W be the price in Risky for one Euro, so that WU and WV are potential claim sizes in (million) Risky from fire claims next year. Suppose that U, V, W are independent and that for $x > 1$

$$\begin{aligned}\mathrm{P}(U \leq x) &= \mathrm{P}(V \leq x) = 1 - x^{-\alpha},\\ \mathrm{P}(W \leq x) &= 1 - x^{-\beta},\end{aligned}$$

where $\alpha > 0$ and $\beta > 0$. Compute the upper tail dependence coefficient $\lambda_U(WU, WV)$ for losses in (million) Risky due to French and German fire claims. (10 p)

LÖSNING TILL TENTAMEN I 5B1580 RISKVÄRDERING OCH RISKHANTERING 2006-06-08

Problem 1

Symmetry and Chebyshev's inequality gives

$$\mathrm{P}(X > x) = \mathrm{P}(|X - 0| > x)/2 \le 10^{-4}/[2x^2]$$

for $x > 0$. Hence,

$$\begin{aligned}
\mathrm{VaR}_{0.99}(X) &= \inf\{x \in \mathbb{R} : \mathrm{P}(X \le x) \ge 0.99\} \\
&= \inf\{x > 0 : \mathrm{P}(X \le x) \ge 0.99\} \\
&= \inf\{x > 0 : \mathrm{P}(X > x) \le 0.01\} \\
&\le \inf\{x > 0 : 10^{-4}/[2x^2] \le 10^{-2}\} \\
&= \inf\{x > 0 : 10^{-2}/[2x^2] \le 1\} \\
&= 1/[10\sqrt{2}] < 1/10.
\end{aligned}$$

Problem 2

The variance of L_A^Δ is

$$\sigma_A^2 = \mathrm{Var}([X_1 - X_2]/2) = \sigma^2(1-\rho)/2$$

and the variance of L_B^Δ is

$$\sigma_B^2 = \mathrm{Var}([X_1 + \cdots + X_n]/n) = \sigma^2[1/n + \rho(n-1)/n] \approx \sigma^2\rho$$

since n is very large. Since the joint distribution of the logreturns is a multivariate t-distribution with 5 degrees of freedom it follows that $L_A^\Delta \stackrel{\mathrm{d}}{=} \sigma_A Z$ and $L_B^\Delta \stackrel{\mathrm{d}}{=} \sigma_B Z$, where Z has a univariate t-distribution (with zero mean and variance one) with 5 degrees of freedom. Hence, $\mathrm{VaR}_{0.99}(L_A^\Delta) > \mathrm{VaR}_{0.99}(L_B^\Delta)$ if and only if $\sigma_A > \sigma_B$. The latter inequality holds (as $n \to \infty$) if $\rho < 1/3$.

Problem 3

$X \leftrightarrow t(3)$, $Y \leftrightarrow N(0,1)$, $Z \leftrightarrow t(10)$, $W \leftrightarrow N(0,\sigma^2)$ with $\sigma > 1$.

Problem 4

(a) The random variables $X_1 \mid Z = z, \dots, X_{100} \mid Z = z$ are independent and Poisson($z/100$)-distributed. Hence, $N \mid Z = z$ is Poisson(z)-distributed. Hence, $\mathrm{P}(N = 1 \mid Z = 1) = 1/e$.

(b)

$$\begin{aligned}
\mathrm{P}(N=1) = E(\mathrm{P}(N=1 \mid Z)) &= \int_0^\infty \mathrm{P}(N=1 \mid Z=z) f_Z(z)\mathrm{d}z \\
&= \int_0^\infty e^z z \frac{z^{\alpha-1}e^{-z/\beta}}{\beta^\alpha\Gamma(\alpha)}\mathrm{d}z \\
&= \int_0^\infty \frac{z^\alpha e^{-z(1+1/\beta)}}{\beta^\alpha\Gamma(\alpha)}\mathrm{d}z \\
&= \int_0^\infty \frac{z^{\widetilde{\alpha}-1}e^{-z/\widetilde{\beta}}}{\beta^\alpha\Gamma(\alpha)}\mathrm{d}z \\
&= \frac{\widetilde{\beta}^{\widetilde{\alpha}}\Gamma(\widetilde{\alpha})}{\beta^\alpha\Gamma(\alpha)} \int_0^\infty \frac{z^{\widetilde{\alpha}-1}e^{-z/\widetilde{\beta}}}{\widetilde{\beta}^{\widetilde{\alpha}}\Gamma(\widetilde{\alpha})}\mathrm{d}z \\
&= \frac{\widetilde{\beta}^{\widetilde{\alpha}}\Gamma(\widetilde{\alpha})}{\beta^\alpha\Gamma(\alpha)},
\end{aligned}$$

where $\widetilde{\beta} = \beta/(1+\beta)$ and $\widetilde{\alpha} = \alpha+1$. Hence,

$$\mathrm{P}(N=1) = \left(\frac{\beta}{1+\beta}\right)^{\alpha+1} \frac{1}{\beta^\alpha}\frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)} = \beta\alpha(1+\beta)^{-\alpha-1} = (1+\beta)^{-1/\beta-1}.$$

Problem 5

Since WU and WV are equally distributed we have

$$\lambda_U(WU, WV) = \lim_{t\to\infty} \mathrm{P}(WU > t, WV > t)/\,\mathrm{P}(WU > t).$$

Set $f(\alpha,\beta,t) = \mathrm{P}(WU > t, WV > t)$ and $g(\alpha,\beta,t) = \mathrm{P}(WU > t)$. Fix $t > 1$.

$$\begin{aligned}
f(\alpha,\beta,t) &= \int_1^\infty \mathrm{P}(wU > t, wV > t)\beta w^{-\beta-1}\mathrm{d}w \\
&= \int_1^\infty \mathrm{P}(wU > t)^2\beta w^{-\beta-1}\mathrm{d}w \\
&= \int_1^t (t/w)^{-2\alpha}\beta w^{-\beta-1}\mathrm{d}w + \int_t^\infty \beta w^{-\beta-1}\mathrm{d}w \\
&= t^{-2\alpha}\int_1^t \beta w^{2\alpha-\beta-1}\mathrm{d}w + t^{-\beta}.
\end{aligned}$$

Hence,

$$f(\alpha,\beta,t) = \begin{cases} 2\alpha t^{-2\alpha}\ln t + t^{-2\alpha}, & \beta = 2\alpha, \\ \frac{2\alpha}{2\alpha-\beta}t^{-\beta} - \frac{\beta}{2\alpha-\beta}t^{-2\alpha}, & \beta \neq 2\alpha. \end{cases}$$

Similarly we have

$$g(\alpha,\beta,t) = t^{-\alpha}\int_1^t \beta w^{\alpha-\beta-1}\mathrm{d}w + t^{-\beta} = \begin{cases} \alpha t^{-\alpha}\ln t + t^{-\alpha}, & \beta = \alpha, \\ \frac{\alpha}{\alpha-\beta}t^{-\beta} - \frac{\beta}{\alpha-\beta}t^{-\alpha}, & \beta \neq \alpha. \end{cases}$$

It follows that

$$\lambda_U(WU, WV) = \lim_{t\to\infty} \frac{f(\alpha,\beta,t)}{g(\alpha,\beta,t)} = \begin{cases} \frac{2\alpha-2\beta}{2\alpha-\beta}, & \beta \in (0,\alpha), \\ 0, & \beta \in [\alpha,\infty). \end{cases}$$

The interpretation is that we have tail dependence if the tail of the exchange rate is heavier than the tail of the claim sizes in their local currencies.