TENTAMEN I 5B1580 RISKVÄRDERING OCH RISKHANTERING FÖR F4 OCH I4 MÅNDAG DEN 18 DECEMBER 2006 KL 14.00–19.00.

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*Tillåtna hjälpmedel*: Inga.

Införda beteckningar skall förklaras och definieras. Resonemang och uträkningar skall vara så utförliga att de är lätta att följa.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

På Matematisk statistiks anslagstavla i entréplanet, Lindstedtsvägen 25, rakt fram innanför porten, kommer det senast den 15 januari 2007 att anslås att rättning och rapportering av tentamensresultaten är klar.

Varje korrekt lösning ger tio poäng. Gränsen för godkänt är preliminärt 25 poäng.

Tentamen kommer att finnas tillgänglig på elevexpeditionen sju veckor efter skrivningstillfället.

Lycka till!

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## Problem 1

You hold a portfolio consisting of a long position of 18 shares of a stock with stock price $S_0 = 140$ SEK today. The ten-day log returns for this stock is assumed to be normally distributed with zero mean and standard deviation 0.01. Let $L$ and $L^{\Delta}$ be the ten-day portfolio loss and the linearized ten-day portfolio loss, respectively. Determine whether $\mathrm{VaR}_{0.99}(L) > \mathrm{VaR}_{0.99}(L^{\Delta})$ or whether $\mathrm{VaR}_{0.99}(L) < \mathrm{VaR}_{0.99}(L^{\Delta})$. (10 p)

## Problem 2

The German investor Dietmar bought one share of a Swedish stock at the Swedish stock market today for $S_0 = 100$ SEK. Dietmar paid in Euro and he bought 1 SEK for $Y_0 = 1/10$ Euro chosing the floating (stochastic) exchange rate.
The German investor Frank also bought one share of this Swedish stock at the Swedish stock market today for $S_0 = 100$ SEK. Also Frank paid in Euro and he bought 1 SEK for 1/8 Euro chosing a fixed (deterministic) exchange rate that does not change during the relevant holding period.
We assume that the log return from today until tomorrow of the stock price $S$ in SEK and the log return from today until tomorrow of the floating exchange rate $Y$

are equally distributed with zero means and a joint elliptical distribution with linear correlation coefficient $\varrho \in (-1, 1)$.
Let $L_{\mathrm{D}}^{\Delta}$ and $L_{\mathrm{F}}^{\Delta}$ be the linearized portfolio losses from today until tomorrow in Euro for Dietmar and Frank, respectively.
Compute $\mathrm{VaR}_{0.99}(L_{\mathrm{D}}^{\Delta})$ and $\mathrm{VaR}_{0.99}(L_{\mathrm{F}}^{\Delta})$ and determine whether it is correct that $\mathrm{VaR}_{0.99}(L_{\mathrm{D}}^{\Delta}) > \mathrm{VaR}_{0.99}(L_{\mathrm{F}}^{\Delta})$ for $\varrho = -0.1$? (10 p)

## Problem 3

Let $(X_1, X_2)$ be a vector of log returns for two stocks for the time period today until tomorrow. Suppose that $X_1$ and $X_2$ are normally distributed with zero means and variances 0.0001 and that $(X_1, X_2)$ has copula $C(u_1, u_2) = (u_1^{-2} + u_2^{-2} - 1)^{-1/2}$ for $(u_1, u_2) \in [0, 1]^2$. Suppose that you have bought one share of each stock today, each for the price 70 SEK. Let $L_1$ and $L_2$ denote the potential losses, from today until tomorrow, for the two stocks.
Compute $\lim_{l \to 70} \mathrm{P}(L_2 > l \mid L_1 > l)$. (10 p)

## Problem 4

Consider the situation in Problem 3.
(a) Which of the plots (upper, middle or lower) in Figure 1 shows a sample from the distribution of $(X_1, X_2)$? Which of the plots shows a sample from the distribution of $(L_1, L_2)$? (2 p)

(b) Compute the empirical estimate of $\mathrm{VaR}_{0.999}$ for your portfolio loss based on the sample shown in Figure 1. (4 p)

(c) Compute the empirical estimate of $\mathrm{ES}_{0.999}$ for your portfolio loss based on the sample shown in Figure 1. (4 p)

## Problem 5

A bank has given loans, each of size 100000 SEK, to two companies A and B. The average interest rate during the next year is modelled by a random variable $Z$ which can take the values 1/100 and 1/10, each with probability 1/2. Conditional on next year's average interest rate $Z$, default during the next year for company A and default during the next year for company B are independent and for both companies default occurs with probability $Z$. In case of a default the bank loses half of the loan size. Compute the expected value of the next year's default loss for the bank (from potential defaults of companies A and B). (10 p)

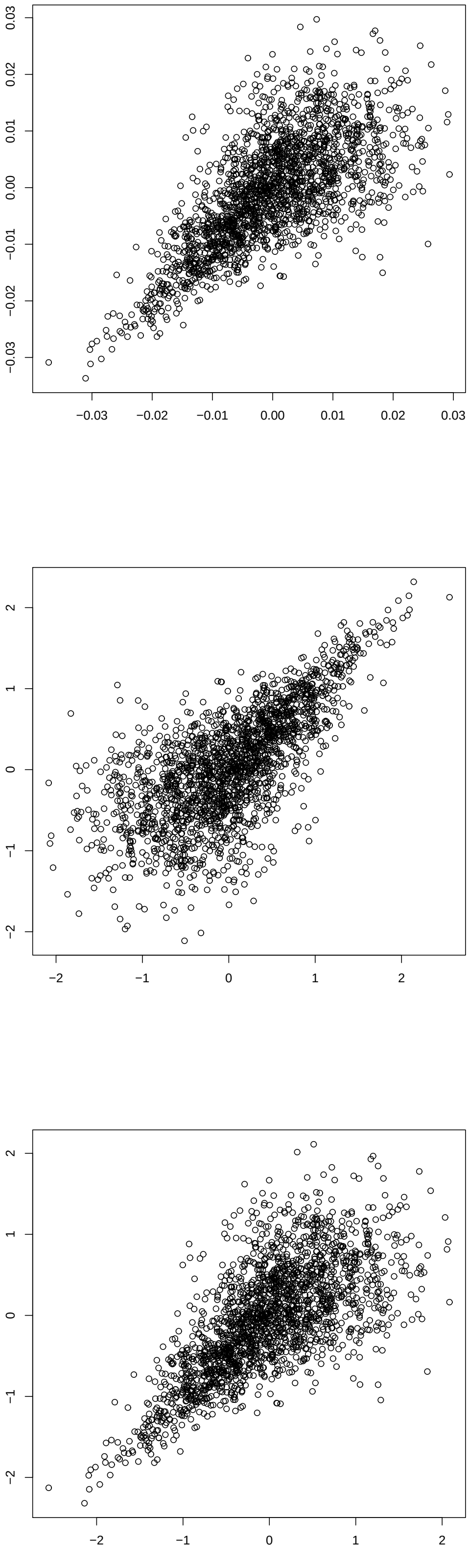


Figure 1: Bivariate scatter plots of samples of size 2000.

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## Problem 1

We have with $C = 140 \cdot 18$ and $X$ denoting the ten-day log return:

$$L = -C(e^X - 1) \text{ and } L^\Delta = -CX.$$

Let $F_L$, $F_{L^\Delta}$ and $F_X$ denote the distribution functions of $L$, $L^\Delta$ and $X$, respectively. Then, for $p \in (0,1)$, one shows that

$$F_L^{-1}(p) = C(1 - e^{F_X^{-1}(1-p)}) \text{ and } F_{L^\Delta}^{-1}(p) = -CF_X^{-1}(1-p).$$

Since the distribution of $X$ is symmetric around 0, this means that

$$F_L^{-1}(p) = C(1 - e^{-F_X^{-1}(p)}) \text{ and } F_{L^\Delta}^{-1}(p) = CF_X^{-1}(p).$$

Notice that $F_X^{-1}(p) > 0$ for $p > 1/2$ and that $F_L^{-1}(p) - F_{L^\Delta}^{-1}(p) = f(F_X^{-1}(p))$ for $f(x) = C(1 - e^{-x} - x)$. Since $f(0) = 0$ and $f'(x) < 0$ for $x > 0$ it follows that $f(x) < 0$ for $x > 0$. Hence, $F_L^{-1}(p) - F_{L^\Delta}^{-1}(p) = f(F_X^{-1}(p)) < 0$ for $p > 1/2$. In particular,

$$\mathrm{VaR}_{0.99}(L) = F_L^{-1}(0.99) < F_{L^\Delta}^{-1}(0.99) = \mathrm{VaR}_{0.99}(L^\Delta).$$

## Problem 2

Let $X_1 = \ln(S_1/S_0)$ and $Z_1 = \ln(Y_1/Y_0)$. Then $(X_1, Z_1) \sim E_2(\boldsymbol{\mu}, \Sigma, \psi)$ with

$$\boldsymbol{\mu} = \mathbf{0}, \quad \Sigma = \begin{pmatrix} \sigma^2 & \sigma^2\varrho \\ \sigma^2\varrho & \sigma^2 \end{pmatrix}.$$

We have

$$\begin{aligned} L_{\mathrm{D}} &= -S_0Y_0\left(\exp\left\{\ln\left(\frac{S_1Y_1}{S_0Y_0}\right) - 1\right\}\right) = -10(\exp\{X_1 + Z_1\} - 1), \\ L_{\mathrm{F}} &= -(S_0/8)(\exp\{X_1\} - 1) = -12.5(\exp\{X_1\} - 1). \end{aligned}$$

Hence,

$$\begin{aligned} L_{\mathrm{D}}^\Delta &= -\begin{pmatrix} 10 & 10 \end{pmatrix}\begin{pmatrix} X_1 \\ Z_1 \end{pmatrix} \stackrel{\mathrm{d}}{=} \left(\begin{pmatrix} 10 & 10 \end{pmatrix}\begin{pmatrix} \sigma^2 & \sigma^2\varrho \\ \sigma^2\varrho & \sigma^2 \end{pmatrix}\begin{pmatrix} 10 \\ 10 \end{pmatrix}\right)^{1/2} W, \\ L_{\mathrm{F}}^\Delta &= -\begin{pmatrix} 12.5 & 0 \end{pmatrix}\begin{pmatrix} X_1 \\ Z_1 \end{pmatrix} \stackrel{\mathrm{d}}{=} \left(\begin{pmatrix} 12.5 & 0 \end{pmatrix}\begin{pmatrix} \sigma^2 & \sigma^2\varrho \\ \sigma^2\varrho & \sigma^2 \end{pmatrix}\begin{pmatrix} 12.5 \\ 0 \end{pmatrix}\right)^{1/2} W, \end{aligned}$$

where $W \sim E_1(0,1,\psi)$. Hence, $\mathrm{VaR}_{0.99}(L_{\mathrm{D}}^\Delta) = \sigma 10\sqrt{2(1+\varrho)}\,\mathrm{VaR}_{0.99}(W)$ and $\mathrm{VaR}_{0.99}(L_{\mathrm{F}}^\Delta) = \sigma 12.5\,\mathrm{VaR}_{0.99}(W)$. If $\varrho = -0.1$, then $10\sqrt{2(1+\varrho)} = 10\sqrt{1.8} > 12.5$ so $\mathrm{VaR}_{0.99}(L_{\mathrm{D}}^\Delta) > \mathrm{VaR}_{0.99}(L_{\mathrm{F}}^\Delta)$.

## Problem 3

It holds that $L_k = f(X_k)$ for $k = 1, 2$ with $f(x) = -70(e^x - 1)$. Notice that $f$ is strictly decreasing. Moreover, $F_X(x)$ is the distribution function of both $X_1$ and and $X_2$ and

$$\begin{aligned} F_L(l) &= \mathrm{P}(L_1 \le l) = \mathrm{P}(f(X_1) \le l) = 1 - F_X(f^{-1}(l)),\\ F_L^{-1}(p) &= f(F_X^{-1}(1-p)) \end{aligned}$$

are the distribution function and quantile function, respectively, of both $L_1$ and $L_2$. Hence, the copula of $(L_1, L_2)$ is given by

$$\begin{aligned} C_{(L_1,L_2)}(u_1,u_2) &= \mathrm{P}(L_1 \le F_L^{-1}(u_1), L_2 \le F_L^{-1}(u_2))\\ &= \mathrm{P}(f(X_1) \le f(F_X^{-1}(1-u_1)), f(X_2) \le f(F_X^{-1}(1-u_2)))\\ &= \mathrm{P}(X_1 \ge F_X^{-1}(1-u_1), X_2 \ge F_X^{-1}(1-u_2))\\ &= \mathrm{P}(F_X(X_1) \ge 1-u_1, F_X(X_2) \ge 1-u_2)\\ &= 1 - (1-u_1) - (1-u_2) + C(1-u_1, 1-u_2)\\ &= C(1-u_1, 1-u_2) + u_1 + u_2 - 1. \end{aligned}$$

Hence,

$$\begin{aligned} \lim_{l\to 70} \mathrm{P}(L_2 > l \mid L_1 > l) &= \lim_{l\to 70} \frac{\mathrm{P}(L_2 > l, L_1 > l)}{\mathrm{P}(L_1 > l)}\\ &= \lim_{l\to 70} \frac{\mathrm{P}(F_L(L_2) > F_L(l), F_L(L_1) > F_L(l))}{\mathrm{P}(F_L(L_1) > F_L(l))}\\ &= \lim_{u\to 1} \frac{1 - 2u + C_{(L_1,L_2)}(u,u)}{1-u}\\ &= \lim_{u\to 1} \frac{C(1-u, 1-u)}{1-u}\\ &= \lim_{u\to 0} \frac{C(u,u)}{u}\\ &\{ \text{ computations like in lecture notes } \}\\ &= 1/\sqrt{2}. \end{aligned}$$

## Problem 4

(a) The upper plot shows a sample from the distribution of $(X_1, X_2)$. The middle plot shows a sample from the distribution of $(L_1, L_2)$.
(b,c) We have $[2000(1 - 0.999)] + 1 = 2 + 1 = 3$. The three largest total losses for $L_1 + L_2$ are given by the pairs $(l_1, l_2) \approx (2.05, 2.15)$, $(l_1, l_2) \approx (2.1, 2.3)$ and $(l_1, l_2) \approx (2.5, 2.1)$. Hence,

$$(l_1 + l_2)_{3,2000} \approx 4.2, \quad (l_1 + l_2)_{2,2000} \approx 4.4, \quad (l_1 + l_2)_{1,2000} \approx 4.6.$$

Hence,

$$\begin{aligned} \widehat{\mathrm{VaR}}_{0.999}(L_1 + L_2) &= (l_1 + l_2)_{3,2000} \approx 4.2 \text{ SEK},\\ \widehat{\mathrm{ES}}_{0.999}(L_1 + L_2) &= \frac{(l_1 + l_2)_{1,2000} + (l_1 + l_2)_{2,2000} + (l_1 + l_2)_{3,2000}}{3} \approx 4.4 \text{ SEK}. \end{aligned}$$

## Problem 5

Let $X_k$ be the default indicator for company $k$ and let $N = X_1 + X_2$ be the total number of defaults. Let $L$ be the total default loss for the bank. $X_1 \mid Z$ and $X_2 \mid Z$ are independent and identically distributed with $\mathrm{P}(X_1 = 1 \mid Z) = Z$ and $\mathrm{P}(X_1 = 0 \mid Z) = 1 - Z$. Hence, $N \mid Z \sim \mathrm{Bin}(2, Z)$ and $L = 50000N$. Hence,

$$\begin{aligned} E(L) &= E(E(L \mid Z)) = 50000E(2Z) = 100000E(Z) \\ &= 100000\left(\frac{1}{2}\frac{1}{100} + \frac{1}{2}\frac{1}{10}\right) = 100000\frac{11}{200} = 5500. \end{aligned}$$

Hence, the expected portfolio default loss is 5500 SEK.