KTH Matematik

TENTAMEN I 5B1580 RISKVÄRDERING OCH RISKHANTERING FÖR F4 OCH I4 TORSDAG DEN 7 JUNI 2007 KL 14.00–19.00.

Examinator: Filip Lindskog, tel. 790 7217, e-post: lindskog@math.kth.se

Tillåtna hjälpmedel: Inga.

Införda beteckningar skall förklaras och definieras. Resonemang och uträkningar skall vara så utförliga att de är lätta att följa.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

På Matematisk statistiks anslagstavla i entréplanet, Lindstedtsvägen 25, rakt fram innanför porten, kommer det senast den 30 juni 2007 att anslås att rättning och rapportering av tentamensresultaten är klar.

Varje korrekt lösning ger tio poäng. Gränsen för godkänt är preliminärt 25 poäng.

Tentamen kommer att finnas tillgänglig på elevexpeditionen sju veckor efter skrivningstillfället.

Lycka till!

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Problem 1

Below you find the 24 insurance claims that exceed 10 Million Kronor out of in total 1949 claims (the total sample size is hence $1949/24 \approx 81$ times larger). The claims are assumed to be observations of independent and identically distributed random variables $X_1, \dots, X_{1949}$.

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 [1] 152.413209  95.168375  41.213000  34.605146  23.191095  18.301611
 [7]  17.569546  16.934801  16.753927  15.527950  15.213358  14.577883
[13]  14.023591  12.701101  12.152200  12.150434  12.059369  11.310395
[19]  11.089682  11.081675  10.692103  10.471204  10.248902  10.125362
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(a) Compute the empirical estimate of $\mathrm{VaR}_{0.99}(X)$ based on all 1949 claims. (3 p)

(b) Recall the POT approximation

$$\mathrm{P}(X > u + x) \approx \frac{N_u}{n}\left(1 + \frac{\widehat{\gamma}}{\widehat{\beta}}x\right)^{-1/\widehat{\gamma}}, \quad x > 0,$$

where n is the total sample size and N_u the number of observations exceeding the threshold u. Use the parameter esimates $(\widehat{\gamma}, \widehat{\beta}) = (0.5, 4.5)$ and the POT approximation above to estimate $\mathrm{VaR}_{0.99}(X)$. (7 p)

Problem 2

Consider the random variable $Z \in (0,1)$ which describes the state of the Swedish economy next year. A bank has given loans to two companies A and B. Conditional on the random variable $Z \in (0,1)$ default of company A next year and default of company B next year are independent and each occur with probability Z. It is assumed that $\mathrm{E}(Z) = 10^{-2}$ and $\mathrm{E}(Z^2) = 10^{-3}$. Consider the default indicator $X_A, X_B \in \{0,1\}$ given by $X_i = 1$ if company i defaults next year and $X_i = 0$ otherwise, for $i = A, B$.

(a) Compute $\mathrm{P}(X_A = 1)$. (3 p)

(b) Compute the linear correlation coefficient for (X_A, X_B). (7 p)

Problem 3

Suppose that the 1-day logreturns of the DAX index are independent and identically distributed (iid) random variables. The estimated mean and standard deviation are $\mu = 2 \cdot 10^{-4}$ and $\sigma = 10^{-2}$. Use the Central limit theorem

$$\frac{S_n - \mathrm{E}(S_n)}{\sqrt{\mathrm{var}(S_n)}} \xrightarrow{d} Z \quad \text{as } n \to \infty, \quad S_n = X_1 + \cdots + X_n,$$

where $X_1, X_2, \ldots$ are iid and Z is standard normally distributed, to estimate $\mathrm{VaR}_{0.95}$ for the 100-day logreturn of the DAX index. Use the fact that $\Phi^{-1}(0.95) \approx 1.64$, where Φ is the standard normal distribution function. (10 p)

Problem 4

Suppose that the total claim amount S one year for an insurance company follows a standard exponential distribution. The insurance company can buy so-called stop-loss reinsurance, for a price p, so that the claim amount exceeding $\mathrm{VaR}_{0.95}(S)$ is paid by the reinsurer. In this case the insurance company has to pay $L = \min(S, \mathrm{VaR}_{0.95}(S)) + p$. Determine for which price p it holds that $\mathrm{VaR}_{0.99}(S) = \mathrm{VaR}_{0.99}(L)$. (10 p)

Problem 5

The random vector (X_1, X_2) has (joint) distribution function

$$\mathrm{P}(X_1 \le x_1, X_2 \le x_2) = \Phi(x_1/\sigma_1)\Phi(x_2/\sigma_2), \quad \sigma_1, \sigma_2 > 0,$$

where Φ is the standard normal distribution function. Compute the expected shortfall ratio $\mathrm{ES}_\alpha(X_1 + X_2)/\,\mathrm{ES}_\alpha(X_1)$. (10 p)

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Problem 1

(a) $[1949(1-0.99)]+1 = 19+1 = 20$. Hence, $\widehat{\mathrm{VaR}}_{0.99}(X) = x_{20,1949} \approx 11.1$.

(b) We get the quantile $F_X^{-1}(p)$ by solving $F_X(F_X^{-1}(p)) = p$ and using the approximation

$$1 - F_X(x) \approx \frac{N_u}{n}\left(1 + \frac{\widehat{\gamma}}{\widehat{\beta}}(x-u)\right)^{-1/\widehat{\gamma}},$$

for $x > u$. This gives

$$F_X^{-1}(p) \approx u + \frac{\widehat{\beta}}{\widehat{\gamma}}\left[\left(\frac{n}{N_u}(1-p)\right)^{-\widehat{\gamma}} - 1\right].$$

With the given values this yields

$$\widehat{\mathrm{VaR}}_{0.99}(X) = 10 + 9[(0.81)^{-1/2} - 1] = 10 + 9[(0.9)^{-1} - 1] = 11.$$

Problem 2

(a) $\mathrm{P}(X_A = 1) = \mathrm{E}(\mathrm{P}(X_A = 1 \mid Z)) = \mathrm{E}(Z) = 0.01$.

(b) Notice that $\mathrm{E}(X_A X_B \mid Z) = \mathrm{P}(X_A = B_B = 1 \mid Z) = Z^2$, $\mathrm{E}(X_A \mid Z) = \mathrm{P}(X_A = 1 \mid Z) = Z$ and

$$\begin{aligned}
\varrho_L(X_A, X_B) &= \frac{\mathrm{E}(X_A X_B) - \mathrm{E}(X_A)\,\mathrm{E}(X_B)}{\sqrt{\mathrm{E}(X_A^2) - \mathrm{E}(X_A)^2}\sqrt{\mathrm{E}(X_B^2) - \mathrm{E}(X_B)^2}} \\
&= \frac{\mathrm{E}(X_A X_B) - \mathrm{E}(X_A)^2}{\mathrm{E}(X_A) - \mathrm{E}(X_A)^2} \\
&= \frac{\mathrm{E}(\mathrm{E}(X_A X_B \mid Z)) - \mathrm{E}(\mathrm{E}(X_A \mid Z))^2}{\mathrm{E}(\mathrm{E}(X_A \mid Z)) - \mathrm{E}(\mathrm{E}(X_A \mid Z))^2} \\
&= \frac{\mathrm{E}(Z^2) - \mathrm{E}(Z)^2}{\mathrm{E}(Z) - \mathrm{E}(Z)^2} \\
&= \frac{0.001 - 0.0001}{0.01 - 0.0001} \\
&= 0.0009/0.0099 \approx 0.09.
\end{aligned}$$

Problem 3

We have

$$Y = \ln(P_{100}/P_0) = \sum_{i=1}^{100} \ln(P_i/P_{i-1}) = \sum_{i=1}^{100} X_i,$$

where the X_i's are iid 1-day logreturns.

$$\mathrm{P}(Y \leq y) = \mathrm{P}\left(\frac{Y - 100\mu}{\sigma\sqrt{100}} \leq \frac{y - 100\mu}{\sigma\sqrt{100}}\right) \approx \Phi\left(\frac{y - 100\mu}{\sigma\sqrt{100}}\right).$$

Assuming that the approximation holds exact and solving

$$\Phi\left(\frac{F_Y^{-1}(p) - 100\mu}{\sigma\sqrt{100}}\right) = p$$

yields $F_Y^{-1}(p) = 100\mu + 10\sigma\Phi^{-1}(p)$. Using the given information yields the estimate $\widehat{\mathrm{VaR}}_{0.95}(Y) = 0.02 + 0.164 = 0.184$.

Problem 4

$\mathrm{VaR}_\alpha(L) = \mathrm{VaR}_{0.95}(S) + p$ for $\alpha \geq 0.95$ (must be shown). Hence, $p = \mathrm{VaR}_{0.99}(S) - \mathrm{VaR}_{0.95}(S)$ gives $\mathrm{VaR}_{0.99}(S) = \mathrm{VaR}_{0.99}(L)$.

Problem 5

We have $(X_1, X_2) \overset{\mathrm{d}}{=} (\sigma_1 Z_1, \sigma_2 Z_2)$, with $(Z_1, Z_2) \sim N_2(\mathbf{0}, \mathbf{I})$. Hence, $X_1 + X_2 \overset{\mathrm{d}}{=} \sqrt{\sigma_1^2 + \sigma_2^2} Z_1$ and

$$\begin{aligned}
\mathrm{ES}_\alpha(X_1 + X_2) &= \mathrm{ES}_\alpha(\sqrt{\sigma_1^2 + \sigma_2^2} Z_1) \\
&= \mathrm{E}(\sqrt{\sigma_1^2 + \sigma_2^2} Z_1 \mid \sqrt{\sigma_1^2 + \sigma_2^2} Z_1 \geq \mathrm{VaR}_\alpha(\sqrt{\sigma_1^2 + \sigma_2^2} Z_1)) \\
&= \mathrm{E}(\sqrt{\sigma_1^2 + \sigma_2^2} Z_1 \mid Z_1 \geq \mathrm{VaR}_\alpha(Z_1)) \\
&= \sqrt{\sigma_1^2 + \sigma_2^2}\, \mathrm{ES}_\alpha(Z_1).
\end{aligned}$$

Similarly, $\mathrm{ES}_\alpha(X_1) = \sigma_1 \,\mathrm{ES}_\alpha(Z_1)$. Hence,

$$\frac{\mathrm{ES}_\alpha(X_1 + X_2)}{\mathrm{ES}_\alpha(X_1)} = \frac{\sqrt{\sigma_1^2 + \sigma_2^2}}{\sigma_1} = \sqrt{1 + (\sigma_2/\sigma_1)^2}.$$