KTH Matematik

EXAMINATION IN SF2980 RISK MANAGEMENT, 2007-12-15, 08.00–13.00.

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*Allowed technical aids*: calculator.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

On January 7, at latest, it will be posted on the notice board of Mathematical Statistics (Lindstedtsvägen 25, straight ahead after passing through the main entrance) that the result of the examination has been reported.

Typically 25 points (out of 50) will be sufficient to pass the exam.

GOOD LUCK!

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## Problem 1

You have a sample $L_1, \dots, L_{250}$ of independent losses with a common, but unknown, distribution $F$ with a continuous distribution function. What is the probability that the empirical Value-at-Risk estimate of $\mathrm{VaR}_{0.99}(F)$ (Value at Risk at confidence level 0.99) based on the sample $L_1, \dots, L_{250}$ exceeds $\mathrm{VaR}_{0.99}(F)$? (10 p)

## Problem 2

Consider a homogeneous loan portfolio consisting of 100 loans. The event that loan $i$ defaults within one year and the event that loan $j$ defaults within one year are dependent for all $i, j \in \{1, \dots, 100\}$. However, conditional on the future value $Z$ of a given macroeconomic variable, defaults for different loans are independent and occur with the probability $Z^\beta$, $\beta > 0$.
Suppose that, for each $i \in \{1, \dots, 100\}$, the probability that loan $i$ defaults within one year is 0.1. Suppose further that $Z$ is uniformly distributed on the interval $(0, 1)$. What is the probability that both loan 1 and loan 17 default within one year. (10 p)

## Problem 3

Suppose that daily log returns of the Noika stock price process are t-distributed with four degrees of freedom, mean $2 \cdot 10^{-4}$ and variance $10^{-4}$. Suppose that you hold a portfolio consisting of (long position) 10 shares of the Noika stock and that the Noika stock price today is 100.

(a) Suppose that the daily log returns are independent. Estimate $\mathrm{VaR}_{0.99}$ for the 250-day portfolio loss. (5 p)

(b) Suppose that the joint distribution for daily log returns for different days is a multivariate t-distribution and that the linear correlation coefficient between any two daily log returns for different days is zero. Estimate $\mathrm{VaR}_{0.99}$ for the 250-day portfolio loss. (5 p)

In (a) and (b) use Figure 1.

## Problem 4

A bank offers you the possibility to invest in a portfolio consisting of a combination of (long positions in) a Canadian and an east European stock market index. The bivariate vector of log returns for the next ten-day period has an elliptical distribution with mean $\mu$ and covariance matrix $\Sigma$ given by

$$\mu = 10^{-3}\begin{pmatrix} 3 \\ 7 \end{pmatrix}, \quad \Sigma = 10^{-4}\begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix},$$

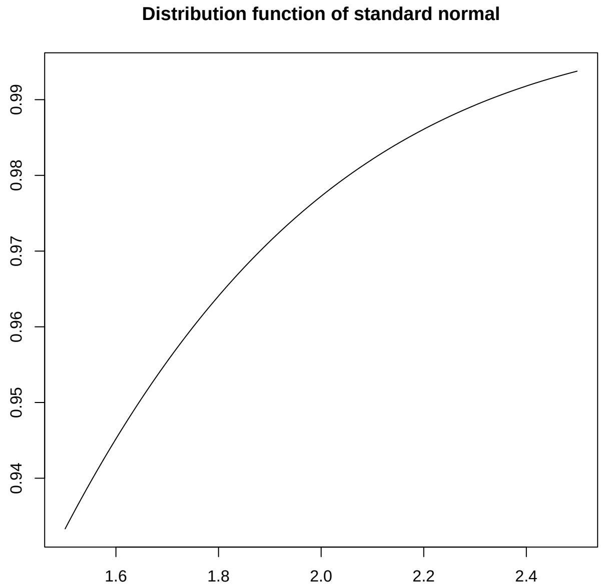
where the first component in the vector corresponds to the Canadian index and the second to the east European index. For both indices, the price today for one share is 100 SEK. The bank holds $50000 = 5 \cdot 10^4$ shares of both indices. The bank reports the risk number $1000000 = 10^6$ SEK, corresponding to $\mathrm{ES}_{0.99}$ (Expected Shortfall at confidence level 0.99) for the linearized loss for its portfolio over the next ten-day period.
What is the risk in terms of $\mathrm{ES}_{0.99}$ for the linearized loss for a portfolio consisting of one share of the Canadian index? (10 p)

## Problem 5

A company is in default one year from today if at that time the debt exceeds the value of its assets. The value of the assets (in units of $10^8$ SEK) for Ericzon, $V_{\mathrm{E}}$, one year from now is assumed to follow a standard exponential distribution. The value of the assets (in units of $10^8$ SEK) for Noika, $V_{\mathrm{N}}$, one year from now is assumed to follow an exponential distribution with mean 2. We assume that the debt for Ericzon $K_{\mathrm{E}}$ and for Noika $K_{\mathrm{N}}$ one year from now are known constants. The vector $(V_{\mathrm{E}}, V_{\mathrm{N}})$ of future asset values has the copula

$$C(u_1, u_2) = \frac{1}{\sqrt{[(u_1^{-2} - 1)^3 + (u_2^{-2} - 1)^3]^{1/3} + 1}}.$$

Both Ericzon and Noika have received the BB rating of Standard and Poors which corresponds to the probability 0.01 of default in one year.
What is the probability that Ericzon will default next year given that Noika will default next year? (10 p)



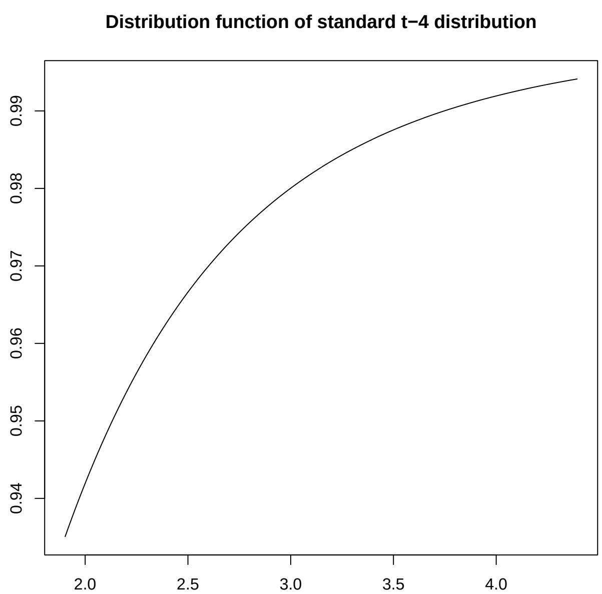




Figure 1: Upper: distribution function of $Z \sim N(0,1)$. Lower: distribution function of a standard $t_4$-distributed random variable $X$ ($X \stackrel{d}{=} \sqrt{4/S}Z$, where $S \sim \chi_4^2$ is independent of $Z$).

SOLUTION TO EXAMINATION IN SF2980 RISK MANAGEMENT 2007-12-15.

## Problem 1

$[250(1-0.99)]+1 = 2+1 = 3$. Hence, $\widehat{\mathrm{VaR}}_{0.99}(F) = L_{3,250}$. Let $F$ also denote the distribution function. Let $Y_{0.99} := \#\{L_i > F^{\leftarrow}(0.99)\}$. We have

$$\begin{aligned} \mathrm{P}(L_i > F^{\leftarrow}(0.99)) &= 1 - \mathrm{P}(L_i \le F^{\leftarrow}(0.99)) \\ &= 1 - \mathrm{P}(F(L_i) \le 0.99) = 0.01. \end{aligned}$$

Hence, $Y_{0.99}$ is Binomial$(250, 0.01)$-distributed.

$$\begin{aligned} &\mathrm{P}(L_{3,250} > F^{-1}(0.99)) = \mathrm{P}(Y_{0.99} \ge 3) = 1 - \mathrm{P}(Y_{0.99} \le 2) \\ &\quad = 1 - \left(\binom{250}{0} 0.01^0 0.99^{250} + \binom{250}{1} 0.01^1 0.99^{249} + \binom{250}{2} 0.01^2 0.99^{248}\right) \\ &\quad \approx 0.457 \end{aligned}$$

## Problem 2

Let $X_i$ be the default indicator for loan $i$. We have

$$\begin{aligned} \mathrm{P}(X_i = 1) &= \mathrm{E}(\mathrm{P}(X_i = 1 \mid Z)) = \mathrm{E}(Z^{\beta}) = 1/(\beta+1), \\ \mathrm{P}(X_1 = 1, X_{17} = 1) &= \mathrm{E}(\mathrm{P}(X_1 = 1 \mid Z)\,\mathrm{P}(X_{17} = 1 \mid Z)) = \mathrm{E}(Z^{2\beta}) = 1/(2\beta+1). \end{aligned}$$

Since $1/(\beta+1) = 1/10$ we have $\beta = 9$. Hence, $\mathrm{P}(X_1 = 1, X_{17} = 1) = 1/19$.

## Problem 3

The 250 future daily log returns are assumed to be $\mathrm{t}_4$-distributed with mean $\mu = 2 \cdot 10^{-4}$ and variance $\sigma^2 = 10^{-4}$. Let $\widetilde{X} = -X$ be the future negative log return over the 250-day period. Then the portfolio loss $L$, its df and its quantile are expressed as

$$\begin{aligned} &L = -1000(e^{-\widetilde{X}} - 1) \quad F_L(l) = F_{\widetilde{X}}(-\ln(1 - l/1000)), \\ &\mathrm{VaR}_p(L) = 1000(1 - \exp\{-\mathrm{VaR}_p(\widetilde{X})\}). \end{aligned}$$

(a) Independent daily log returns. By the central limit theorem, approximately, $\widetilde{X} \sim N(-250\mu, 250\sigma^2)$. Hence,

$$\begin{aligned} &\mathrm{VaR}_{0.99}(\widetilde{X}) \approx -250\mu + \sqrt{250}\sigma\Phi^{-1}(0.99) \approx -0.05 + \sqrt{0.025} \cdot 2.33 \approx 0.3184053, \\ &\mathrm{VaR}_{0.99}(L) \approx 272.6920. \end{aligned}$$

(b) The joint distribution of the daily log returns is a 250-dimensional $\mathrm{t}_4$-distribution with linear correlation zero between distinct components. $\widetilde{X} \stackrel{\mathrm{d}}{=} -250\mu + \sqrt{250/2}\sigma Z$, where $Z$ has a standard $t$ distribution with four degrees of freedom. Hence,

$$\begin{aligned} &\mathrm{VaR}_{0.99}(\widetilde{X}) \approx -250\mu + \sqrt{250/2}\sigma t_4^{-1}(0.99) \approx -0.05 + \sqrt{0.0125} \cdot 3.75 \approx 0.3692627, \\ &\mathrm{VaR}_{0.99}(L) \approx 308.7562. \end{aligned}$$

## Problem 4

Let $X$ be the bivariate log return vector. We have $X \stackrel{d}{=} \mu + AY$, where $AA^T = \Sigma$ and $Y$ has a bivariate spherical distribution. Let $L^\Delta$ be the linearized portfolio loss. Then (see lecture notes)

$$L^\Delta = -w^T X \stackrel{d}{=} -w^T\mu + \sqrt{w^T \Sigma w} Y_1.$$

Hence,

$$\mathrm{ES}_{0.99}(L^\Delta) = \mathrm{ES}_{0.99}(-w^T X) = -w^T\mu + \sqrt{w^T\Sigma w}\,\mathrm{ES}_{0.99}(Y_1),$$

where $\mathrm{ES}_{0.99}(L^\Delta) = 10^6$, $w^T = 5 \cdot 10^6(1,1)$, $w^T\mu = 5 \cdot 10^4$ and $\sqrt{w^T\Sigma w} = 1.5 \cdot 10^5$. This gives $\mathrm{ES}_{0.99}(Y_1) = 7$. Hence, the linearized loss $\widetilde{L}^\Delta$ of the portfolio with one share of the Canadian stock market index we have with $\widetilde{w} = (100, 0)$

$$\mathrm{ES}_{0.99}(\widetilde{L}^\Delta) = -\widetilde{w}^T\mu + \sqrt{\widetilde{w}^T\Sigma\widetilde{w}}\,\mathrm{ES}_{0.99}(Y_1) = -0.3 + \sqrt{2} \cdot 7 \approx 9.6.$$

## Problem 5

With $F_\mathrm{E}, F_\mathrm{N}$ denoting the dfs of $V_\mathrm{E}, V_\mathrm{N}$ we have

$$\begin{aligned}
\mathrm{P}(V_\mathrm{E} < K_\mathrm{E} \mid V_\mathrm{N} < K_\mathrm{N}) &= \frac{\mathrm{P}(V_\mathrm{E} < K_\mathrm{E}, V_\mathrm{N} < K_\mathrm{N})}{\mathrm{P}(V_\mathrm{N} < K_\mathrm{N})} \\
&= \frac{\mathrm{P}(V_\mathrm{E} < F_\mathrm{E}^{-1}(0.01), V_\mathrm{N} < F_\mathrm{N}^{-1}(0.01))}{\mathrm{P}(V_\mathrm{N} < F_\mathrm{N}^{-1}(0.01))} \\
&= \frac{\mathrm{P}(F_\mathrm{E}(V_\mathrm{E}) < 0.01, F_\mathrm{N}(V_\mathrm{N}) < 0.01)}{\mathrm{P}(F_\mathrm{N}(V_\mathrm{N}) < 0.01)} \\
&= \frac{C(0.01, 0.01)}{0.01} = 100[(2 \cdot 9999^3)^{1/3} + 1]^{-1/2} \approx 0.89.
\end{aligned}$$