KTH Matematik

EXAMINATION IN SF2980 RISK MANAGEMENT, 2008-06-04, 14:00–19:00.

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*Allowed technical aids*: calculator.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

GOOD LUCK!

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## Problem 1

You have bought 10 shares of a stock with share price \$1 today and with historical daily log-returns shown in the qq-plot in Figure 1. For a standard t-distributed random variable $Z$ with 4 degrees of freedom it holds that $\mathrm{ES}_{0.99}(Z) = 5.22$. Estimate $\mathrm{ES}_{0.99}$ for your linearized portfolio loss from today until tomorrow. (10 p)

## Problem 2

Let $X = (X_1, \dots, X_d)^{\mathrm{T}}$ be normally distributed. The classical minimum variance portfolio is given by the vector $w = (w_1, \dots, w_d)^{\mathrm{T}}$ of portfolio weights that solves the optimization problem

$$\min_{w \in \mathcal{W}_r} \mathrm{var}(w^{\mathrm{T}} X). \tag{1}$$

where $r$ is a predetermined fixed number and the minimization is over the set $\mathcal{W}_r$ of weight vectors $w = (w_1, \dots, w_d)^{\mathrm{T}} \in \mathbb{R}^d$ satisfying $\mathrm{E}(w^{\mathrm{T}} X) = r$ and $w_1 + \dots + w_d = 1$. Since the variance is a questionable choice of risk measure we want to replace it by a risk measure $\varrho$ with the properties:

$$\varrho(Z + a) = \varrho(Z) + a \text{ for } a \in \mathbb{R} \text{ and } \varrho(\lambda Z) = \lambda^{3/2} \varrho(Z) \text{ for } \lambda \geq 0.$$

Show that the portfolio weights we get if we use $\varrho$ instead of the variance in (1) are the same as for the minimum variance portfolio. (10 p)

## Problem 3

Consider a homogeneous loan portfolio consisting of 2 loans, during a one-year period. The default probability for the two loans are the same. The event that loan 1 defaults and the event that loan 2 defaults are dependent. However, conditional on the future value $Z$ of a given macroeconomic variable, uniformly distributed on the interval $(0, 1)$, defaults for different loans are independent and occur with the probability $\beta Z$, $\beta \in (0, 1)$. Suppose that the probability that both loans default is 0.03. What is the probability that no loan defaults? (10 p)

## Problem 4

You hold a portfolio consisting of long positions in two different assets, one unit of the first asset and two units of the second asset.
Historical daily asset prices, in $ per unit, of the two assets until and including today (column 20 shows today's prices, column 1 shows the prices 19 days ago):

```
      [,1] [,2]  [,3]  [,4]  [,5]  [,6] [,7] [,8]  [,9] [,10] [,11] [,12]
[1,] 81.35 80.4 81.05 83.35 83.00 83.30 86.0 85.5 84.50    84 84.05 82.35
[2,] 81.00 81.5 81.50 81.85 81.25 81.45 83.5 83.5 83.75    86 85.75 84.60
     [,13]  [,14] [,15] [,16] [,17] [,18] [,19] [,20]
[1,] 83.45 83.50  84.4  86.9 85.90 82.55 83.75 84.75
[2,] 83.85 84.55  84.0  84.3 84.75 85.35 87.00 85.75
```

Historical daily log-returns of the two assets during this time period:

```
             [,1]         [,2]         [,3]         [,4]         [,5]
[1,]  0.003694585 -0.011746657  0.008052072  0.027982361  -0.004208001
[2,] -0.018349139  0.006153866  0.000000000  0.004285284  -0.007357483
             [,6]        [,7]         [,8]         [,9]         [,10]
[1,]  0.003607941  0.03189875  -0.00583092  -0.011764842  -0.005934736
[2,]  0.002458514  0.02485730   0.00000000   0.002989539   0.026511126
            [,11]       [,12]        [,13]        [,14]         [,15]
[1,]  0.000595061 -0.02043340  0.013269193  0.0005989817  0.010720770
[2,] -0.002911210 -0.01350182 -0.008904778  0.0083135871 -0.006526277
            [,16]        [,17]        [,18]       [,19]         [,20]
[1,]  0.029190631 -0.011574203 -0.039779659  0.01443200    0.01186958
[2,]  0.003565066  0.005323881  0.007054703  0.01914767   -0.01447203
```

Use historical simulation to estimate $\mathrm{VaR}_{0.98}$ for the linearized portfolio loss from today until tomorrow. (10 p)

## Problem 5

Consider the following simulation algorithm, where $A$ is the matrix

$$A = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ 0 & 1 \end{pmatrix}.$$

- Simulate 2 independent random variates $Z_1, Z_2$ from $N(0, 1)$.
- Simulate 2 independent random variates $S_1, S_2$ from $\chi^2_3$ (Chi-square distribution with 3 degrees of freedom) independent of $Z_1, Z_2$.
- Set $(Y_1, Y_2)^{\mathrm{T}} = A(Z_1, Z_2)^{\mathrm{T}}$.
- Set $X_k = \frac{\sqrt{3}}{\sqrt{S_k}} Y_k$ for $k = 1, 2$.

The algorithm is repeated 2000 times to produce the random sample

$$\{(X_1^1, X_2^1), \ldots, (X_1^{2000}, X_2^{2000})\}.$$

One of the plots in Figure 2 is generated from this sample. Which one? (an answer without explanation will not be accepted) (10 p)

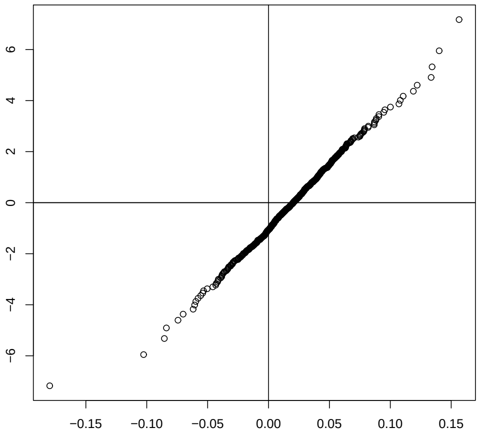


Figure 1: qqplot of log-returns (x-axis) against quantiles of a standard t-distribution with 4 degrees of freedom (y-axis).

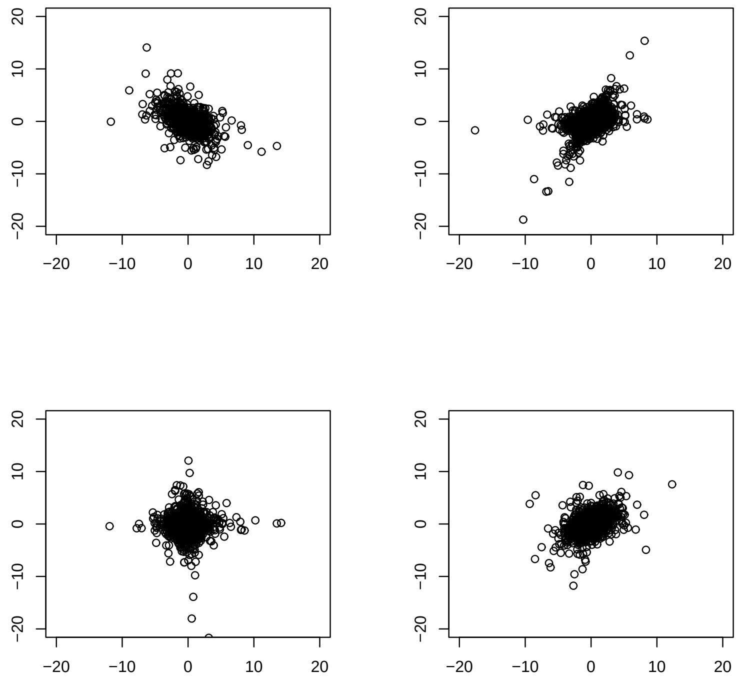


Figure 2: Plots of bivariate samples.

SOLUTION TO EXAMINATION IN SF2980 RISK MANAGEMENT 2008-06-04.

## Problem 1

Let $X$ be a standard t-distributed rv with four degrees of freedom, with df $F$. Let $Y$ the log-return from today until tomorrow, with df $G$. We see that $F^{-1}(q) = a + bG^{-1}(q)$, for $q \in (0,1)$, with $a \approx -1$ and $b \approx 50$. Hence, approximatively, $G^{-1}(q) = F^{-1}(q)/50 + 1/50$. Hence, we may assume that $Y \stackrel{\mathrm{d}}{=} X/50 + 1/50$. This means that

$$\mathrm{ES}_{0.99}(L^{\Delta}) = \mathrm{ES}_{0.99}(-10Y) = -1/5 + \mathrm{ES}_{0.99}(X)/5 \approx 0.844(\$).$$

## Problem 2

We have $X \stackrel{\mathrm{d}}{=} \mu + AY$, where $AA^{\mathrm{T}} = \Sigma$ and $\mathbf{Y} \sim N_d(0, I)$. Hence,

$$w^{\mathrm{T}}X \stackrel{\mathrm{d}}{=} w^{\mathrm{T}}\mu + w^{\mathrm{T}}AY = w^{\mathrm{T}}\mu + (A^{\mathrm{T}}w)^{\mathrm{T}}Y \stackrel{\mathrm{d}}{=} w^{\mathrm{T}}\mu + \sqrt{w^{\mathrm{T}}\Sigma w}Y_1.$$

Hence, for $w \in \mathcal{W}_r$,

$$\varrho(w^{\mathrm{T}}X) = w^{\mathrm{T}}\mu + (w^{\mathrm{T}}\Sigma w)^{3/4}\varrho(Y_1) = r + \mathrm{var}(w^{\mathrm{T}}X)^{3/4}\varrho(Y_1)$$

Hence, the $w$ minimizing $\varrho(w^{\mathrm{T}}X)$ is the same as the $w$ minimizing $\mathrm{var}(w^{\mathrm{T}}X)$.

## Problem 3

Let $X_i$ be the default indicator for loan $i$. We have

$$\begin{aligned}
\mathrm{P}(X_1 = 1, X_2 = 1) &= \mathrm{E}(\mathrm{P}(X_1 = 1, X_2 = 1 \mid Z)) = \mathrm{E}(\mathrm{P}(X_1 = 1 \mid Z)\,\mathrm{P}(X_2 = 1 \mid Z)) \\
&= \mathrm{E}(\beta^2 Z^2) = \beta^2/3 \\
\mathrm{P}(X_1 = 0, X_2 = 0) &= \cdots = \mathrm{E}((1-\beta Z)^2) = 1 - \beta + \beta^2/3.
\end{aligned}$$

Hence, $\beta = 0.3$ and it follows that $\mathrm{P}(X_1 = 0, X_2 = 0) = 0.73$.

## Problem 4

The current portfolio consists of (long positions) one unit of the first asset and two units of the second asset. In terms of the log-returns $X_1$ and $X_2$, the linearized portfolio loss is $L^{\Delta} = -(84.75X_1 + 2 \cdot 85.75X_2)$. We have $[20(1-0.98)] + 1 = 1$. Therefore the estimate of $\mathrm{VaR}_{0.98}(L^{\Delta})$ is $l^{\Delta}_{1,20} = \$4.047293$ (column nr 12).

## Problem 5

The upper-right plot. The two independent $\chi^2_3$-distributed variables means that this is not a bivariate $t_3$-distribution and not an elliptical distribution. The linear correlation coefficient is $1/2$ and the variances are equal. Hence only the upper-right and the lower-right plot can be the result of the simulation. However, the lower-right plot shows an elliptical distributed.