KTH Matematik

EXAMINATION IN SF2980 RISK MANAGEMENT, 2009-06-07, 14:00–19:00.

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Allowed technical aids: calculator.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

GOOD LUCK!

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Problem 1

Let $Z_1, Z_2, \dots, Z_{1000}$ be historical daily negative log-returns of an asset. That is, if S_n is the price of the asset on day n, then $Z_n = -(\log S_n - \log S_{n-1})$. Let $Z_{1:1000} \geq Z_{2:1000} \geq \cdots \geq Z_{1000:1000}$ be the ordered sample and suppose you have computed

$$
\begin{array}{llll}
Z_{1:1000} = 0.0350, & Z_{2:1000} = 0.0332, & Z_{3:1000} = 0.0318, & Z_{4:1000} = 0.0302, \\
Z_{5:1000} = 0.0296, & Z_{6:1000} = 0.0214, & Z_{7:1000} = 0.0200, & Z_{8:1000} = 0.0188, \\
Z_{993:1000} = -0.0194, & Z_{994:1000} = -0.0196, & Z_{995:1000} = -0.0214, & Z_{996:1000} = -0.0221, \\
Z_{997:1000} = -0.0244, & Z_{998:1000} = -0.0252, & Z_{999:1000} = -0.0290, & Z_{1000:1000} = -0.0335,
\end{array}
$$

and

$$\overline{Z} = \frac{1}{1000}\sum_{i=1}^{1000} Z_i = 5.8446 \times 10^{-5}, \qquad s^2 = \frac{1}{999}\sum_{i=1}^{1000}(Z_i - \overline{Z})^2 = 4.8373 \times 10^{-5}.$$

Suppose the asset price today is $S = 100$. Compute the 0.995-quantile for the one-day loss of a long position in the asset using

(a) the empirical method, (5 p)

(b) the variance-covariance method. (5 p)

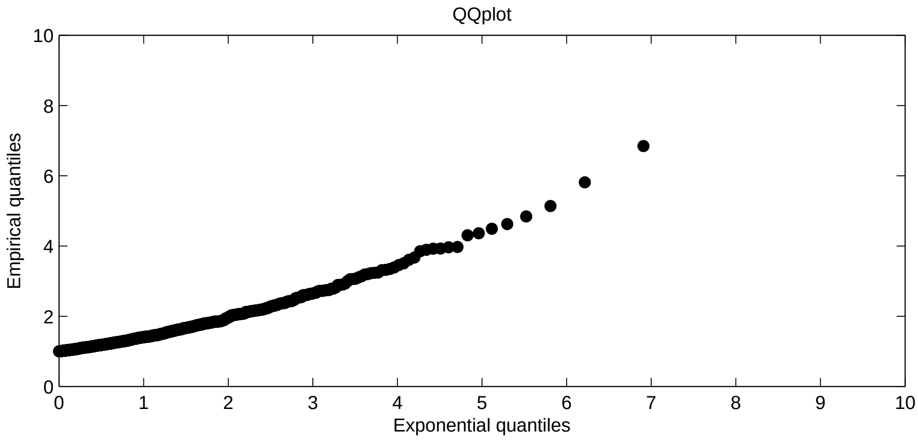

Figure 1: QQ-plot for empirical quantiles of 1000 insurance claims (y-axis) against standard exponential quantiles (x-axis).

Problem 2

A QQ-plot of 1000 historical insurance claims against a standard exponential distribution is given in Figure 1.
(a) Give an estimate for the 0.95-quantile of the claim size distribution. (5 p)

(b) Give an estimate for the 0.95-expected shortfall of the claim size distribution, where, for a given distribution F the p-expected shortfall is defined as

$$\frac{1}{1-p}\int_p^1 F^{-1}(u)du.$$

(5 p)

Problem 3

Consider the scatter plot in Figure 2 of daily log-returns for two assets. The results of fitting location-scale t-distributions to the marginal distribution of each assets log-returns are given in Table 1.
Below are three model suggestions. For each model you should argue if that model is appropriate for the observed data set. Moreover, you should argue which of the three models that you find most appropriate for the observed data set. You must motivate your answer.

1. Multivariate t-distribution with 4 degrees of freedom and covariance matrix given by

$$\Sigma = 10^{-4} \times \begin{pmatrix} 0.037 & 0.0318 \\ 0.0318 & 0.412 \end{pmatrix}.$$

2. Marginal distributions given by the fitted $t_{2.9}$-location-scale and $t_{5.03}$-location-scale in Table 1 and a t_4-copula with parameter $\varrho = 0.25$.

3. Marginal distributions given by by the fitted $t_{2.9}$-location-scale and $t_{5.03}$-location-scale in Table 1 and a Gaussian copula with parameter $\varrho = 0.25$.

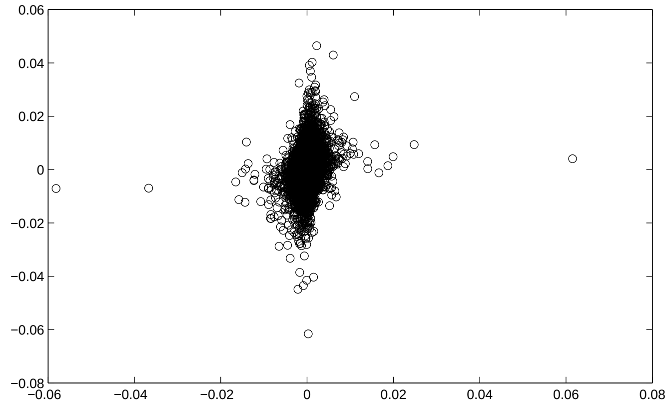

Figure 2: Scatterplot of daily log-returns for two assets. Log-returns of asset 1 on x-axis and log-returns of asset 2 on y-axis.

Marginal 1		Parameter	Estimate
Distribution:	t location-scale	mu	5.62574×10^{-6}
Log likelihood:	51334	sigma	0.000994136
Mean:	5.62574×10^{-6}	nu	2.94617
Variance:	3.07738×10^{-6}		
Marginal 2		Parameter	Estimate
Distribution:	t location-scale	mu	-5.66385×10^{-6}
Log likelihood:	36764.4	sigma	0.00497764
Mean:	-5.66385×10^{-6}	nu	5.02664
Variance:	4.11494×10^{-5}		

Table 1: Output of marginal fitting. A t_ν-location-scale distribution is fitted to each marginal. Recall X is t_ν-location scale if $X = \mu + \sigma Z$ where Z is standard t_ν.

(10 p)

Problem 4

The joint yearly log-returns of two assets is modeled using a normal variance mixture with mean vector and covariance matrix

$$\mu = 10^{-2} \times \begin{pmatrix} 4 \\ 8 \end{pmatrix}, \qquad \Sigma = 10^{-4} \times \begin{pmatrix} 8 & 1 \\ 1 & 16 \end{pmatrix}.$$

The price of one share of each asset is 100. The 0.99-quantile for the linearized loss resulting from holding a long position in asset 1 over the one-year time horizon is 6.6. Compute the 0.99-quantile for the one year linearized loss of a portfolio consisting of one share of each asset. (10 p)

Problem 5

Consider a latent variable model for a homogoenous portfolio of n credit risk. Let p be the default probability for each company, let $Y, Y_1, Y_2, \dots$ be iid $N(0,1)$ random variables, and ϱ be a parameter. Let the default indicator X_i, $i = 1, \dots, n$, for each obligor be given by

$$X_i = \begin{cases} 1, & \text{for } \sqrt{\varrho}Y + \sqrt{1-\varrho}Y_i \leq \Phi^{-1}(p), \\ 0, & \text{otherwise.} \end{cases}$$

Here Φ is the standard normal distribution function.
(a) Find a random variable Θ (expressed in terms of the given variables) such that the default indicators are conditionally iid $\mathrm{Ber}(\theta)$ given $\Theta = \theta$, (5 p)

(b) and show that the following formula holds for the α-quantile of Θ;

$$F_\Theta^{-1}(\alpha) = \Phi^{-1}(\alpha)\frac{\sqrt{\varrho}}{\sqrt{1-\varrho}} + \Phi^{-1}(p)\frac{1}{\sqrt{1-\varrho}}.$$

(5 p)

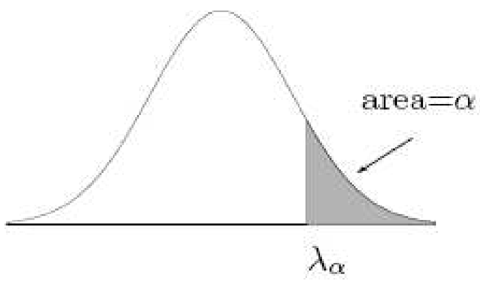

Figure 3: Standard normal quantiles $P(X > \lambda_\alpha) = \alpha$ where $X \sim N(0, 1)$.

α	λ_α
0.1	1.2816
0.05	1.6449
0.025	1.9600
0.01	2.3263
0.005	2.5758
0.001	3.0902
0.0005	3.2905
0.0001	3.7190

Some common distributions:

Exponential distribution: $\text{Exp}(\lambda)$, $F(x) = 1 - e^{-\lambda x}$.

t-location-scale, $t_\nu(\mu, \sigma)$: if $Z \sim t_\nu$ has a standard Student t-distribution with ν degrees of freedom, then $X = \mu + \sigma Z$ has $t_\nu(\mu, \sigma)$-distribution.

SOLUTION TO EXAMINATION IN SF2980 RISK MANAGEMENT 2010-06-07.

Problem 1

(a) Since $F_Z^{-1}(0.995) = Z_{6:1000} = 0.0214$ and $L = g(Z)$ with $g(z) = S(1-e^{-z})$ which is increasing and continuous it follows that

$$F_L^{-1}(0.995) = F_{g(Z)}^{-1}(0.995) = g(F_Z^{-1}(0.995)) = 100(1 - e^{-0.0214}) = 2.1173.$$

(b) With the variance-covariance method the linearized loss is $L^{\Delta} = SZ$. Then

$$\begin{aligned}F_L^{-1}(0.995) &\approx S\Big(\overline{Z} + \sqrt{s^2}\Phi^{-1}(0.995)\Big)\\ &= 100(5.8446 \times 10^{-5} + 0.0070 \times 2.5758) = 1.7974.\end{aligned}$$

Problem 2

(a) The distribution function of the standard exponential is $F(x) = 1-e^{-x}$ and hence the quantile function is $F^{-1}(p) = -\log(1-p)$. Then $F^{-1}(0.95) = -\log 0.05 \approx 3$ and from the QQ-plot we see that the corresponding empirical quantile is about 2.8.
(b) Approximating the tail of QQ-plot by a linear function we find that $F_{emp}^{-1}(u)$ is approximately equal to $-\log(1-u)$ for $u \geq 0.95$. Then it follows that

$$\begin{aligned}\frac{1}{0.05}\int_{0.95}^{1} F_{emp}^{-1}(u)du &\approx \frac{1}{0.05}\int_{0.95}^{1} -\log(1-u)du\\ &= \frac{1}{0.05}[(1-u)\log(1-u) + u]_{0.95}^{1}\\ &= 1 - \log 0.05 \approx 4.\end{aligned}$$

Problem 3

1. The multivariate t_4 is an elliptical distribution, but the sample does not appear to have elliptical symmetry. In particular, the extreme values appear to be close to the coordinate axis.
2. The t-location-scale marginals probably gives a good marginal fit. The t_4-copula has tail-dependence, and with marginals which are rather close to t_4-location scale the resulting distribution is not far from that in item 1. Moreover, the t_4-copula has tail-dependence, which does not seem present in the data.
3. The Gaussian copula has no tail-dependence, which is consistent with the data. Moreover, the fitted t-location-scale marginals probably gives a good marginal fit. This is the most appropriate distribution for this data set.

Problem 4

Write the joint log-returns as $X \stackrel{d}{=} \mu + WAZ$ where $AA^T = \Sigma$ and $Z \sim N(0, I)$. Then the loss $L_1 \stackrel{d}{=} 100(-\mu_1 + \sqrt{\sigma_{11}}WZ_1)$ and

$$6.6 = F_{L_1}^{-1}(0.99) = 100(-\mu_1 + \sqrt{\sigma_{11}}F_{WZ_1}^{-1}(0.99)).$$

This implies $F^{-1}_{WZ_1}(0.99) = 3.7469$ and for $L = -100(11)X$ the 0.99-quantile is given by

$$\begin{aligned} F_L^{-1}(0.99) &= 100(-\mu_1 - \mu_2 + \sqrt{\sigma_{11} + 2\sigma_{12} + \sigma_{22}} F^{-1}_{WZ_1}(0.99) \\ &= 7.11. \end{aligned}$$

Problem 5

(a) Since the event $\{X_i = 1\}$ is the same as

$$\Big\{ Y_i \leq \frac{\Phi^{-1}(p)}{\sqrt{1-\varrho}} - \frac{\sqrt{\varrho}Y}{\sqrt{1-\varrho}} \Big\}.$$

and $Y_i \sim N(0,1)$ it follows that, with

$$\Theta = \Theta(Y) = \Phi\Big(\frac{\Phi^{-1}(p)}{\sqrt{1-\varrho}} - \frac{\sqrt{\varrho}Y}{\sqrt{1-\varrho}}\Big)$$

we have

$$\begin{aligned} &P(X_1 = x_1, \ldots, X_n = x_n \mid \Theta = \theta) \\ &\quad = P(\cap_{i:X_i=1}\{\Phi(Y_i) \leq \theta\} \cap_{i:X_i=0} \{\Phi(Y_i) \geq \theta\} \mid \Theta(Y) = \theta) \\ &\quad = \{ \text{ independence } \} \\ &\quad = \prod_{i:x_i=1} P(\Phi(Y_i) \leq \theta) \prod_{i:x_i=0} P(\Phi(Y_i) \geq \theta) \end{aligned}$$

Since $\Phi(Y_i)$ is uniform the above is equal to

$$\theta^{x_1+\cdots+x_n}(1-\theta)^{n-(x_1+\cdots+x_n)},$$

which shows that default indicators are conditionally iid Ber(θ) given $\Theta = \theta$.
(b) The distribution function is given by

$$\begin{aligned} F_\Theta(\theta) = P(\Theta \leq \theta) &= P\Big[\Phi\Big(\frac{\Phi^{-1}(p)}{\sqrt{1-\varrho}} - \frac{\sqrt{\varrho}Y}{\sqrt{1-\varrho}}\Big) \leq \theta\Big] \\ &= P\Big[Y \geq \frac{\Phi^{-1}(p)}{\sqrt{\varrho}} - \frac{\Phi^{-1}(\theta)\sqrt{1-\varrho}}{\sqrt{\varrho}}\Big] \\ &= \Phi\Big(- \frac{\Phi^{-1}(p)}{\sqrt{\varrho}} + \frac{\Phi^{-1}(\theta)\sqrt{1-\varrho}}{\sqrt{\varrho}}\Big) \end{aligned}$$

Then,

$$F_\Theta^{-1}(\alpha) = \Phi^{-1}(\alpha)\frac{\sqrt{\varrho}}{\sqrt{1-\varrho}} + \Phi^{-1}(p)\frac{1}{\sqrt{1-\varrho}}.$$