KTH Mathematics

Examination in SF2980 Risk Management, December 17, 2010, 14:00–19:00.

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Allowed technical aids: calculator.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

Use Figure 2 for standard Normal probabilities and quantiles.

Good luck!

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Problem 1

Adjusted closing prices for Dow Jones for the past 21 days and today. Today's closing price appears as the last element below.

```
10751.27 10944.72 10967.65 10948.58 11006.48 11010.34 11020.40 11096.08
11096.92 11062.78 11143.69 10978.62 11107.97 11146.57 11132.56 11164.05
11169.46 11126.28 11113.95 11118.40 11124.62 11188.72
```

The corresponding daily percentage returns for Dow Jones, in the same order as above.

```
1.0179932 1.0020951 0.9982613 1.0052884 1.0003507 1.0009137 1.0068673
1.0000757 0.9969235 1.0073137 0.9851871 1.0117820 1.0034750 0.9987431
1.0028286 1.0004846 0.9961341 0.9988918 1.0004004 1.0005594 1.0057620
```

Adjusted closing prices for Nasdaq for the past 21 days and today. Today's closing price appears as the last element below.

```
2344.52 2399.83 2380.66 2383.67 2401.91 2402.33 2417.92 2441.23 2435.38
2468.77 2480.66 2436.95 2457.39 2459.67 2479.39 2490.85 2497.29 2503.26
2507.37 2507.41 2504.84 2533.52
```

The corresponding daily percentage returns for Nasdaq, in the same order as above.

```
1.0235912 0.9920119 1.0012644 1.0076521 1.0001749 1.0064895 1.0096405
0.9976037 1.0137104 1.0048162 0.9823797 1.0083875 1.0009278 1.0080173
1.0046221 1.0025855 1.0023906 1.0016419 1.0000160 0.9989750 1.0114498
```

Use the historical simulation approach to estimate $\mathrm{VaR}_{0.05}(X)$, where X is the difference between the value tomorrow and the value today of a portfolio consisting of a long position of one share of the Dow Jones index and a long position of two shares of the Nasdaq index. (10 p)

Problem 2

Today you buy a European call option on the value of the Dow Jones index six months from now. The strike price of the option is 11200, the value of the index now 11189, the option price is 783 and the Black-Scholes implied volatility of the option is 0.25. Estimate $\mathrm{VaR}_{0.05}(X)$, where X is the difference between the price of the option tomorrow and the price of the option now.

The Black-Scholes formula for the price C_0 at time 0 of a call option with strike price K maturing at time T with implied volatility σ_0 reads:

$$C_0 = S_0\Phi(d_1) - Ke^{-r_0T}\Phi(d_2),$$
$$d_1 = \frac{\log(S_0/K) + (r_0 + \sigma_0^2/2)T}{\sigma_0\sqrt{T}} \text{ and } d_2 = d_1 - \sigma_0\sqrt{T},$$

where S_0 is the time 0 spot price of the underlying asset which is assumed not to pay any dividends, time is measured in years and 0 means now. You may ignore varying interest rates and set $r_0 = 0$. The partial derivative of the option price now with respect to the current spot price (the Black-Scholes delta) is $\Phi(d_1)$. The partial derivative of the option price now with respect to the current implied volatility (the Black-Scholes vega) is $\phi(d_1)S_0\sqrt{T}$. (Φ and ϕ are the distribution function and density for the standard Normal distribution.)
Assume that the log-return from today until tomorrow for Dow Jones has variance $4 \cdot 10^{-4}$, that the change in implied volatility for the option from today until tomorrow has variance $9 \cdot 10^{-4}$, and that their joint distribution is a bivariate Normal distribution with zero mean and linear correlation coefficient -0.5. (10 p)

Problem 3

Consider the random vector (X_1, X_2) which has the dependence structure (or copula) of a bivariate standard Student's t distribution with three degrees of freedom and linear correlation coefficient 0. X_1 is Normally distributed with mean 0.03 and variance 0.04 and X_2 is Student's t distributed with four degrees of freedom, mean 0.02 and variance 0.09.
We may write $(X_1, X_2) \stackrel{d}{=} g(Z_1, Z_2, Z_3, Z_4, Z_5)$, where $Z_1, \dots, Z_5$ are independent and standard Normally distributed. Determine the function $g : \mathbb{R}^5 \to \mathbb{R}^2$ expressed in terms of the standard Normal distribution function Φ and the distribution function t_ν of the standard Student's t distribution with ν degrees of freedom (for appropriate values of ν). (10 p)

Problem 4

Consider the plots in Figure 1. The plot to the left shows the claim sizes of the 1000 fire claims during 10 years for an insurance company specializing in fire insurance. The empirical estimates of the mean and standard deviation for the claim size distribution are 0.97 and 1.98, respectively. The plot to the right shows a qq-plot the excesses over the threshold 3.346 (50 data points) against the quantiles of the GPD

$$G(x) = 1 - \left(1 + \frac{\hat{\gamma}}{\hat{\beta}}x\right)^{-1/\hat{\gamma}},$$

with ML-estimated parameters $(\hat{\gamma}, \hat{\beta}) = (0.47, 2.12)$.
Suppose that next year there will be 100 fire claims out of 10000 insured. The insurance company wants to estimate the amount of capital K it needs at the end of next year so that the probability that the total claim amount exceeds this capital K is q.

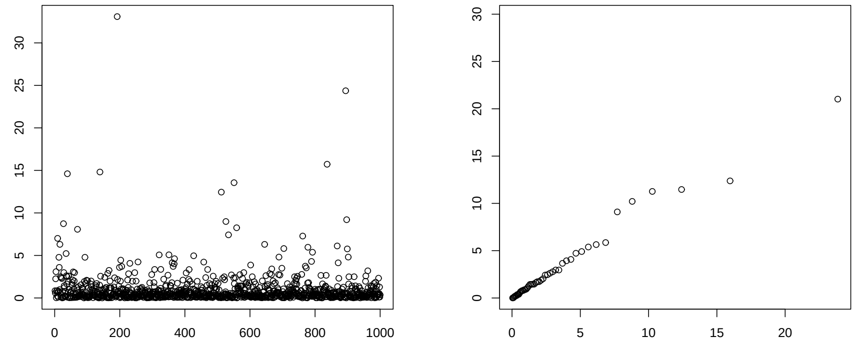

Figure 1: Plot of the fire claims data and a qq-plot of the excess data (y-axis) against quantiles of the GPD with estimated parameters (x-axis).

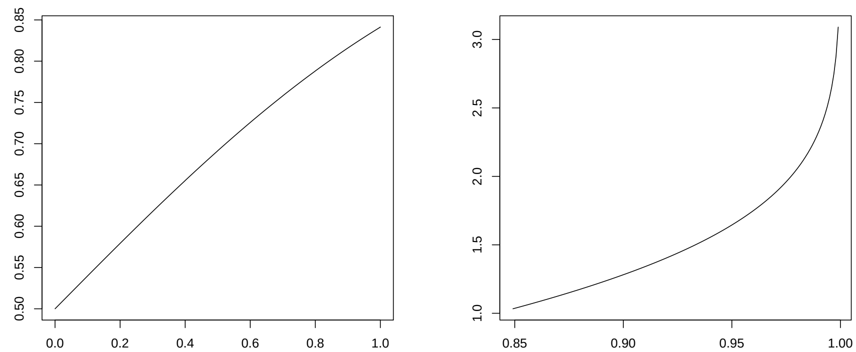

Figure 2: Standard Normal distribution function (left) and quantile function (right).

(a) Suppose that $q = 0.1$. Estimate K. (5 p)

(b) Suppose that $q = 0.001$. Estimate K. (5 p)

Problem 5

Today you take a long position of 100 shares of a stock market index. The current value of one share of the index is 100 and you intend to hold this position one year. To finance this position you take a one-year loan at the risk-free one-year interest rate. This interest rate is determined by the price 0.96 of a risk-free one-year zero-coupon bond that pays 1 one year from now.

The estimates of the quantiles $F_Z^{-1}(0.01)$ and $F_Z^{-1}(0.99)$ for the one-year log-return Z of the index are -0.51 and 0.65, respectively. You may assume that F_Z is continuous and strictly increasing.

Estimate the smallest amount of money that if added to your position now and invested in a risk-free zero coupon bond ensures that the probability of a strictly negative value for your net worth one year from now is not greater than 0.01. (10 p)

Problem 1

We set

$$L_k = V_0 - V_1(R_k^{\mathrm{DJ}}, R_k^{\mathrm{N}}) = 11188.72(1 - R_k^{\mathrm{DJ}}) + 2 \cdot 2533.52(1 - R_k^{\mathrm{N}})$$

for $k = 1, \dots, 21$ which gives

-320.858603	17.034651	13.047847	-97.943270	-4.809939	-43.105707
-125.684786	11.295302	-35.048649	-106.234782	255.019843	-174.325365
-43.581874	-26.561183	-55.069245	-18.522598	31.141235	4.079830
-4.560772	-1.065816	-122.486106			

We observe that the biggest loss correspond to the 11th pair of percentage returns and the second biggest loss correspond to the 17th pair of percentage returns. The estimate of $\mathrm{VaR}_{0.05}(X)$ is $l_{[21\cdot 0.05]+1,21} = l_{2,21} = 31.141235$.

Problem 2

Let $s_1 = 0.02$, $s_2 = 0.03$, and $T = 0.5$. We get

$$\begin{aligned} C - C_0 &\approx C_0 + \frac{\partial C_0}{\partial S_0}(S - S_0) + \frac{\partial C_0}{\partial \sigma_0}(\sigma - \sigma_0) - C_0 \\ &\approx \frac{\partial C_0}{\partial S_0} S_0 Z_1 + \frac{\partial C_0}{\partial \sigma_0} Z_2 \\ &= S_0 \Phi(d_1) Z_1 + S_0 \phi(d_1)\sqrt{T} Z_2 \\ &\stackrel{\mathrm{d}}{=} S_0 \Phi(d_1) s_1 Y_1 + S_0 \phi(d_1) \sqrt{T} s_2 Y_2, \end{aligned}$$

where Y_1, Y_2 are standard Normally distributed with a joint Normal distribution and linear correlation $\rho = -0.5$. Since $a_1 Y_1 + a_2 Y_2 \stackrel{\mathrm{d}}{=} (a_1^2 + a_2^2 + 2a_1 a_2 \rho)^{1/2} Y_1$ and $\mathrm{VaR}_{0.05}(cY_1) = c\Phi^{-1}(0.95)$ we get, with $d_1 \approx 0.083$, $\Phi(d_1) \approx 0.533$, and $\phi(d_1) \approx 0.398$,

$$\begin{aligned} \mathrm{VaR}_{0.05}(C - C_0) &\approx S_0 \Big(\Phi(d_1)^2 s_1^2 + \phi(d_1)^2 T s_2^2 - \Phi(d_1)\phi(d_1)\sqrt{T} s_1 s_2 \Big)^{1/2} \Phi^{-1}(0.95) \\ &\approx 179.2526. \end{aligned}$$

We may compare this result to the result for holding one share of the index:

$$\mathrm{VaR}_{0.05}(S - S_0) \approx S_0 s_1 \Phi^{-1}(0.95) \approx 368.0853.$$

Problem 3

The random vector

$$\Big(\frac{3}{Z_1^2 + Z_2^2 + Z_3^2} \Big)^{1/2} (Z_4, Z_5)$$

has a Student's t distribution with three degrees of freedom. Therefore

$$g(z_1, \dots, z_5) = (a(z_1, z_2, z_3, z_4), b(z_1, z_2, z_3, z_5)),$$

where

$$a(z_1,z_2,z_3,z_4) = 0.03 + 0.2\Phi^{-1}\Big(t_3\Big(\frac{3}{z_1^2+z_2^2+z_3^2}\Big)^{1/2} z_4\Big)\Big),$$
$$b(z_1,z_2,z_3,z_5) = 0.02 + \frac{0.3}{\sqrt{2}}t_4^{-1}\Big(t_3\Big(\frac{3}{z_1^2+z_2^2+z_3^2}\Big)^{1/2} z_5\Big)\Big).$$

Problem 4

Set $S_{100} := X_1 + \cdots + X_{100}$, where $X_1, \ldots, X_{100}$ are iid insurance claim sizes for next year. For $q = 0.1$ the Central Limit Theorem approximation is appropriate:

$$q = \mathrm{P}(S_{100} > K) = \mathrm{P}\left(\frac{S_{100} - 100\mu}{\sqrt{100}\sigma} > \frac{K - 100\mu}{\sqrt{100}\sigma}\right) \approx 1 - \Phi\Big(\frac{K-100\mu}{\sqrt{100}\sigma}\Big)$$

which gives $K \approx 100\mu + \sqrt{100}\sigma\Phi^{-1}(1-q) \approx 122$.
For $q = 0.001$ the heavy-tail approximation seems more appropriate:

$$q = \mathrm{P}(S_{100} > K) \approx 100\,\mathrm{P}(X_1 > K).$$

The POT method gives the estimate

$$q \approx 100\,\mathrm{P}(X_1 > K) = 100\,\mathrm{P}(X_1 > u + K - u) \approx 100\frac{N_u}{n}\Big(1 + \widehat{\gamma}\frac{K-u}{\widehat{\beta}}\Big)^{-1/\widehat{\gamma}}$$

which gives, with $N_u/n = 0.05$ and $u = 3.346$,

$$K \approx 3.346 + \frac{\widehat{\beta}}{\widehat{\gamma}}\Big(0.0002^{-\widehat{\gamma}} - 1\Big) \approx 246.$$

The corresponding estimate using CLT is 158. The empirical estimates from a sample of size 10^6 were 129 and 424. In particular we note that for $q = 0.1$ the CLT estimate is accurate whereas for $q = 0.001$ the heavy-tail approximation for the sum together with the extrapolation using the GPD is perhaps acceptable but none of the estimates are accurate for $q = 0.001$.

Problem 5

Set $B_0 = 0.96$, $V_0 = 100 \cdot 100$, and let V be the value of the index position one year from now. What we are looking for is an estimate of $\mathrm{VaR}_{0.01}(V - V_0/B_0)$.

$$\begin{aligned}\mathrm{VaR}_{0.01}(V - V_0/B_0) &= V_0 + \mathrm{VaR}_{0.01}(V) = V_0 + F^{-1}_{-B_0V}(1-0.01)\\ &= V_0 - B_0F_V^{-1}(0.01) = V_0 - B_0V_0\exp\{F_Z^{-1}(0.01)\}\\ &\approx 10000(1 - 0.96\exp\{-0.51\}) \approx 4235.242.\end{aligned}$$

Linearization, $e^z \approx 1 + z$, is not appropriate here as since -0.51 as too far from 0.