KTH Mathematics

Examination in SF2980 Risk Management, June 8, 2011, 14:00–19:00.

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Allowed technical aids: calculator.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

Good luck!

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Problem 1

A bank uses the empirical Value-at-Risk estimator $\widehat{\mathrm{VaR}}_{n,p}(X) = L_{[np]+1,n}$ to estimate $\mathrm{VaR}_p(X)$, where $L_{1,n} \geq \cdots \geq L_{n,n}$ are the ordered discounted losses based on a sample of (supposedly) independent and identically distributed loss variables $L_1, \ldots, L_n$ whose common distribution function is assumed to be continuous and strictly increasing.

Suppose that $p = 0.01$ and compute the probability

$$\mathrm{P}\left(\widehat{\mathrm{VaR}}_{n,p}(X) > \mathrm{VaR}_p(X)\right)$$

for sample sizes $n = 100$ and $n = 199$. (10 p)

Problem 2

Today you enter an interest rate swap agreement. Every six months, starting in six months, until an including three years from today, you will pay a floating interest on the notional principal $L = 1'000'000$ dollars. In return you will receive fixed interest rate payments at these dates. Today the value of fixed interest rate payments of the swap is

$$cL \sum_{k=1}^{6} \exp\Big\{ -\frac{k}{2} r_{k/2} \Big\},$$

where r_t is the current zero-coupon bond rate for maturity in t years. The value of the floating interest rate payments is

$$L\Big(1 - \exp\{-3r_3\}\Big).$$

The constant c is set so that the two values are equal today. The current zero-coupon bond rates are $r_{k/2} = k/2 \cdot 10^{-2}$ per year.

Suppose that the change $\Delta\mathbf{r} = (\Delta r_{0.5}, \ldots, \Delta r_3)$ in the above zero-coupon bond rates from today until tomorrow can be modeled by a 6-dimensional Normal distribution with zero mean. Suppose further (to simplify the problem) that the linear correlation coefficient $\mathrm{Cor}(\Delta r_s, \Delta r_t) = 1$ for any pair of one-day changes in zero-coupon bond rates, and that $\mathrm{Var}(\Delta r_t) = 25 \cdot 10^{-8}$ for all t.

Estimate $\mathrm{VaR}_{0.05}(V)$, where V is the value tomorrow of your swap position. (10 p)

Problem 3

Today you buy a European call option on the value of the Dow Jones index six months from now. The strike price of the option is 11200, the value of the index now 11189, the option price is 783 and the Black-Scholes implied volatility of the option is 0.25.
The Black-Scholes formula for the price C_0 at time 0 of a call option with strike price K maturing at time T with implied volatility σ_0 reads:

$$C_0 = S_0\Phi(d_1) - Ke^{-r_0T}\Phi(d_2),$$
$$d_1 = \frac{\log(S_0/K) + (r_0 + \sigma_0^2/2)T}{\sigma_0\sqrt{T}} \text{ and } d_2 = d_1 - \sigma_0\sqrt{T},$$

where S_0 is the time 0 spot price of the underlying asset which is assumed not to pay any dividends, time is measured in years and 0 means now. You may ignore varying interest rates and set $r_0 = 0$. The partial derivative of the option price now with respect to the current spot price (the Black-Scholes delta) is $\Phi(d_1)$. The partial derivative of the option price now with respect to the current implied volatility (the Black-Scholes vega) is $\phi(d_1)S_0\sqrt{T}$. (Φ and ϕ are the distribution function and density for the standard Normal distribution.)
Assume that the log-return from today until tomorrow for Dow Jones, denoted by Z_1, has variance $4 \cdot 10^{-4}$, that the change in implied volatility for the option from today until tomorrow, denoted by Z_2, has variance $9 \cdot 10^{-4}$, and that their joint distribution is a bivariate Normal distribution with zero mean and linear correlation coefficient -0.5.
Express the value tomorrow of the call option in terms of Z_1 and Z_2. (10 p)

Problem 4

Consider the plots in Figure 1.

(a) The plot to the left shows the quantiles of claim sizes of 1000 insurance claims (y-axis) against the quantiles of a Gamma distribution (x-axis). With f being the density function of the Gamma distribution, estimate the density g of the claim size distribution in terms of f. Motivate properly! (5 p)

(b) The plot to the right shows the quantiles of a log-normal distribution (y-axis) against the quantiles of a Gamma distribution (x-axis). If F denotes the log-normal distribution function and G denotes the Gamma distribution function, what can you say about the limit $\lim_{x\to\infty}(1 - F(x))/(1 - G(x))$? Motivate properly! (5 p)

Problem 5

Figure 2 shows samples of size 1000 from bivariate distributions.

(a) Let (U, V) be a random vector corresponding to the upper left scatter plot. Suggest a bivariate distribution function for (U, V). Motivate properly! (5 p)

(b) Which of the scatter plots correspond to the random vector $(\Phi^{-1}(U), t_3^{-1}(V))$, where Φ and t_3 are the distribution functions of standard Normal and Student's t with three degrees of freedom, respectively. Motivate properly! (5 p)

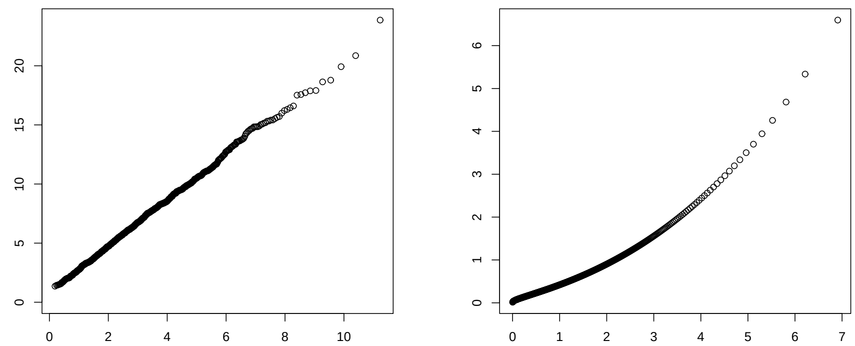

Figure 1: Left: QQ-plot of insurance claims (y-axis) against quantiles of a Gamma distribution (x-axis). Right: QQ-plot of log-normal (y-axis) against a Gamma distribution (x-axis).

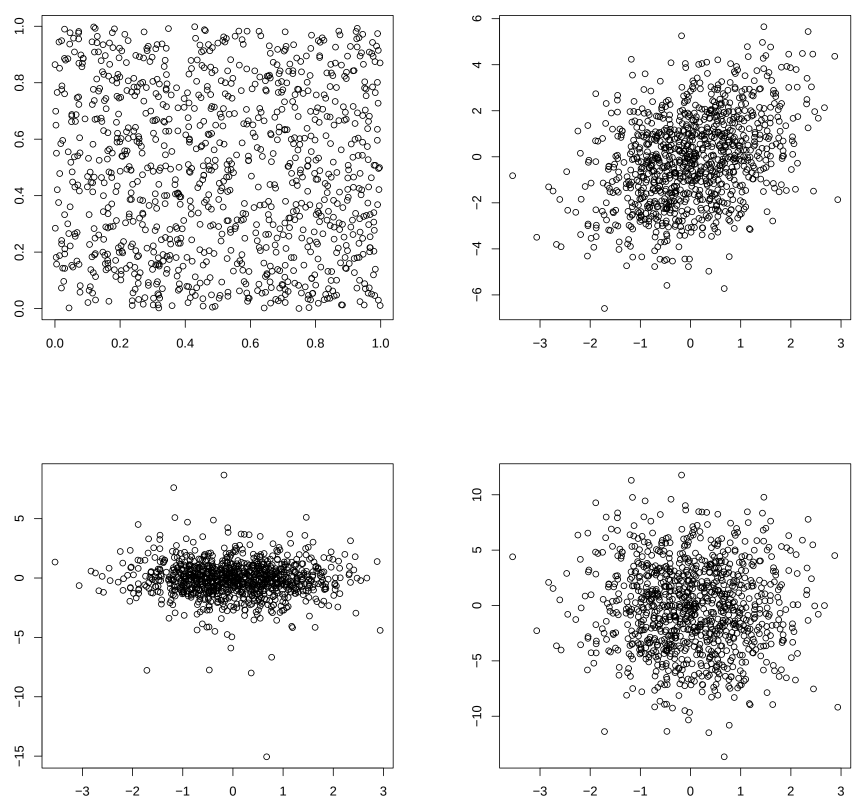

Figure 2: Samples of size 1000 from bivariate distributions.

Problem 1

Write Y for the number of L_ks that exceed $\mathrm{VaR}_p(X) = F_L^{-1}(1-p)$ and note that Y is $\mathrm{Bin}(n,p)$-distributed.

$$\begin{aligned}\mathrm{P}(L_{2,170} > F_L^{-1}(0.99)) &= \mathrm{P}(Y \geq 2) = 1 - \mathrm{P}(Y=0) - \mathrm{P}(Y=1) \\ &= 1 - 0.01^0 \cdot 0.99^n - n \cdot 0.01^1 \cdot 0.99^{n-1} \\ &\approx \begin{cases} 0.26 & n = 100, \\ 0.59 & n = 199. \end{cases}\end{aligned}$$

Problem 2

Set

$$g(\mathbf{r}) = cL\sum_{k=1}^{6} \exp\Big\{-\frac{k}{2}r_{k/2}\Big\} - L\Big(1-\exp\{-3r_3\}\Big).$$

Then

$$\begin{aligned}\frac{\partial g}{\partial r_{k/2}}(\mathbf{r}) &= -cL\frac{k}{2}\exp\Big\{-\frac{k}{2}r_{k/2}\Big\}, \quad k = 1,\dots,5, \\ \frac{\partial g}{\partial r_3}(\mathbf{r}) &= -L(c+1)3\exp\{-3r_3\}.\end{aligned}$$

Using linearization we have

$$g(\mathbf{r}_0 + \Delta\mathbf{r}) \approx g(\mathbf{r}_0) + \nabla g(\mathbf{r}_0)^{\mathrm{T}}\Delta\mathbf{r} \sim \mathrm{N}(0, \nabla g(\mathbf{r}_0)^{\mathrm{T}}\Sigma\nabla g(\mathbf{r}_0)),$$

where Σ is the covariance matrix of $\Delta\mathbf{r}$. Since all pairwise correlations are one, here we have

$$\Delta\mathbf{r} \stackrel{\mathrm{d}}{=} 5 \cdot 10^{-4} \cdot (Z,\dots,Z),$$

where $Z \sim \mathrm{N}(0,1)$. In particular,

$$\nabla g(\mathbf{r}_0)^{\mathrm{T}}\Delta\mathbf{r} \stackrel{\mathrm{d}}{=} 5\cdot 10^{-4}\Big(\sum_{k=1}^{6}\frac{\partial g}{\partial r_{k/2}}(\mathbf{r}_0)\Big)Z =: 5\cdot 10^{-4}KZ.$$

We have

$$c = \frac{1-\exp\{-9/100\}}{\sum_{k=1}^{6}\exp\{-(k/2)^2/100\}} \approx 0.01489225$$

and

$$K = -L\Big\{c\Big(0.5e^{-0.5^2/100} + \cdots + 3e^{-3^2/100}\Big) + 3e^{-3^2/100}\Big\} = -2890230.$$

Therefore,

$$\begin{aligned}\mathrm{VaR}_{0.05}(V) &= \mathrm{VaR}_{0.05}(\nabla g(\mathbf{r}_0)^{\mathrm{T}}\Delta\mathbf{r}) \approx 5\cdot 10^{-4}|K|\,\mathrm{VaR}_{0.05}(-Z) \\ &= 5\cdot 10^{-4}|K|\Phi^{-1}(0.95) \approx 2377.002 \text{ dollars.}\end{aligned}$$

Problem 3

Let C be the random value of the call option tomorrow.

$$C = S_0 e^{Z_1} \Phi(d_1) - K e^{-r_0 T'} \Phi(d_2),$$
$$d_1 = \frac{Z_1 + \log(S_0/K) + (r_0 + (\sigma_0 + Z_2)^2/2)T'}{(\sigma_0 + Z_2)\sqrt{T'}} \text{ and } d_2 = d_1 - (\sigma_0 + Z_2)\sqrt{T'},$$

where $T' \approx T - 1/250$.

Problem 4

(a) The qq-plot looks linear and the line $y = 2x+1$ fits well (should be verified). This means that $G^{-1}(p) \approx 2F^{-1}(p) + 1$, where G and F are the distribution functions of the claim size distribution and Gamma distribution, respectively. Therefore, $G(x) \approx F((x-1)/2)$ which implies that $g(x) \approx f((x-1)/2)/2$.

(b) The qq-plot curves up to the right. This means that the quantile function $F^{-1}(p)$ grows faster than $G^{-1}(p)$ as p tends to 1. Therefore, $1 - F(x)$ decays slower than $1 - G(x)$ as x tends to infinity. Therefore, $\lim_{x\to\infty}(1 - F(x))/(1 - G(x)) = \infty$.

Problem 5

(a) Looks like a uniform distribution on $[0,1]^2$, i.e. the distribution of (U, V) with U, V uniformly distributed on $[0,1]$ and independent. $\mathrm{P}(U \le u, V \le v) = uv$ for u, v in $[0,1]$.

(b) The lower left plot. The t_3-distribution has much heavier tails than the Normal distribution. From the lower left scatter plot we see that the second component has much heavier left and right tails, whereas the other scatter plots show samples from bivariate random vectors with similar, light tails (the probability mass decays rather fast).