KTH Mathematics

Examination in SF2980 Risk Management, June 11, 2012, 14:00–19:00.

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Allowed technical aids and literature: calculator, the printed course literature (Risk and portfolio analysis: Part I & Part II), the printed errata.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow. Only an answer, without an explanation, will give 0 points. You are supposed to make clever use of the figures presented below. The effects of interest rates over short time periods (a few days) may be ignored.

GOOD LUCK!

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The Black-Scholes formula for the price of a European put option is given by

$$p(S,K,\sigma,r,T) = Ke^{-rT}\Phi(-d_2) - S\Phi(-d_1),$$
$$d_1 = \frac{\log(S/K) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T},$$

where Φ denotes the standard Normal distribution function. Moreover,

$$\frac{\partial}{\partial S}p(S,K,\sigma,r,T) = \Phi(d_1) - 1, \quad \frac{\partial}{\partial \sigma}p(S,K,\sigma,r,T) = S\phi(d_1)\sqrt{T}.$$

It holds that $t_{3.9}^{-1}(0.99) \approx 3.8$, where $t_{3.9}$ is the distribution function of the Student's t distribution with parameter triplet $(\mu,\sigma,\nu) = (0,1,3.9)$.

Problem 1

Consider a portfolio consisting of 1000 British Telecom shares and 1000 European put options with maturity in three months with strike price 180 GBP on the value of a British Telecom share. The current share price of British Telecom is 185.5 GBP and the British Telecom share does not pay dividends during the lifetime of the option. Assume that the interest rate is constant and equal to 2% for all maturities, that the implied volatility of the put option is 20% per year and stays constant, and that the British Telecom log-return is Student's t distributed with parameter triplet $(\mu,\sigma,\nu) = (0, 0.014, 3.9)$. Compute $\mathrm{VaR}_{0.01}(V_1 - V_0)$ in GBP, where V_k is the portfolio value in GBP k days from now. Do not use linearization. (10 p)

Problem 2

Consider the portfolio and the assumptions in Problem 1. Use linearization to estimate $\mathrm{VaR}_{0.01}(V_1 - V_0)$ in GBP, where V_k is the portfolio value in GBP k days from now. (10 p)

Problem 3

Consider the portfolio in Problem 1. Assume that the interest rate is constant and equal to 2% for all maturities, that the implied volatility of the put option is 20% per year, that the one-day log-return of the implied volatility has zero mean and standard deviation 0.015, and that the pair of log-returns for the share price and implied volatility has a Student's t distribution with uncorrelated components. Assume that the British Telecom log-return is Student's t distributed with parameter triplet $(\mu, \sigma, \nu) = (0, 0.014, 3.9)$. The current share price of British Telecom is 185.5 GBP. Estimate $\mathrm{VaR}_{0.01}(V_1 - V_0)$ in GBP, where V_k is the portfolio value in GBP k days from now. (10 p)

Problem 4

Consider the portfolio and the assumptions in Problem 1. Use Figure 1 to estimate empirically $\mathrm{VaR}_{0.01}(V_1 - V_0)$ in SEK, where V_k is the portfolio value in SEK k days from now. The current SEK/GBP exchange rate is 0.0942 (the amount of GBP one can buy for one unit of SEK). (10 p)

Problem 5

The lower plot in Figure 1 shows a simulated sample of size 500 from the random vector (X, Y) with independent components. One of the components has distribution function $1 - e^{-x/4}$ and the other one has distribution function $1 - x^{-2}$.

(a) Estimate $\mathrm{P}(X + Y > 15)$. (5 p)

(b) Estimate $\mathrm{P}(X + Y > 40)$. (5 p)

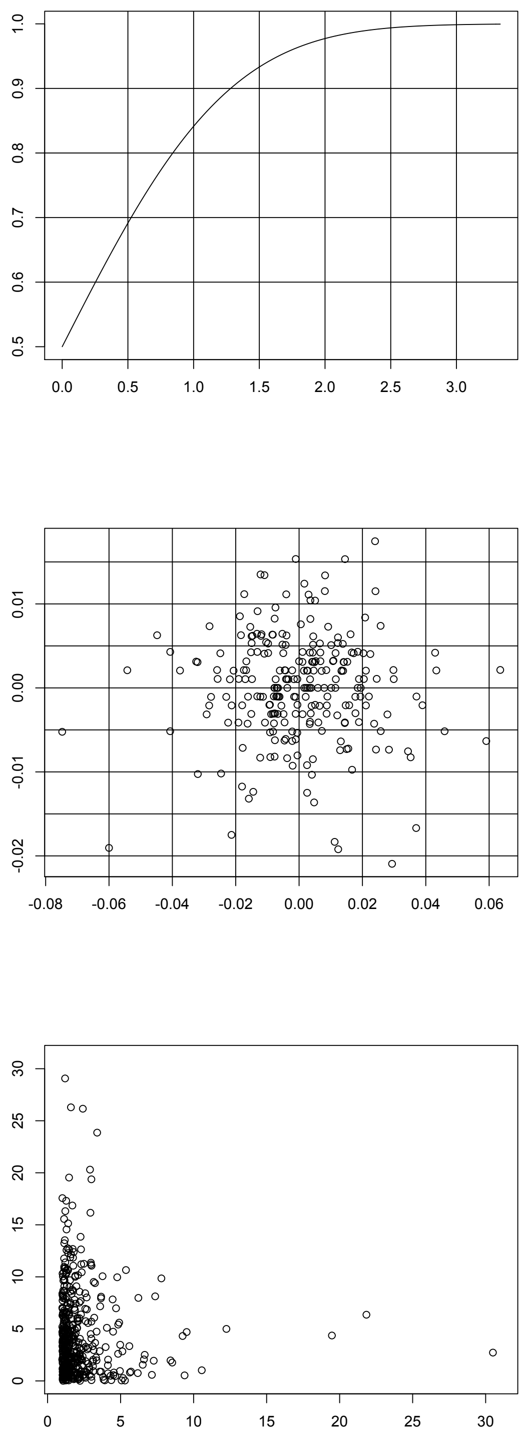

Figure 1: Upper plot: the standard Normal distribution function. Middle plot: scatter plot of 249 log-return pairs: British Telecom in GBP on the x-axis and the SEK/GBP exchange rate on the y-axis. Lower plot: sample of size 500 from a bivariate distribution with Exponential and Pareto marginal distributions.

Problem 1

$V_1 = 1000(S_1 + p(S_1))$, where p is the Black-Scholes put option formula with $\sigma = 0.2, K = 180, r = 0.02, T = 1/4$. Write $V_1 = g(S_1)$ and notice that g is continuous and strictly increasing.

$$\begin{aligned}\mathrm{VaR}_{0.01}(V_1 - V_0) &= V_0 + F^{-1}_{-V_1/R_0}(0.99)\\ &= V_0 - \frac{1}{R_0}F^{-1}_{g(S_1)}(0.01)\\ &= V_0 - \frac{1}{R_0}g(F^{-1}_{S_1}(0.01)).\end{aligned}$$

Here $R_0 \approx 1$, $V_0 \approx 4.513659$, and

$$\begin{aligned}F^{-1}_{S_1}(0.01) &= S_0 \exp\{F^{-1}_X(0.01)\}\\ &= S_0 \exp\{\sigma_X t^{-1}_\nu(0.01)\}\\ &\approx 175.8879.\end{aligned}$$

Therefore, $\mathrm{VaR}_{0.01}(V_1 - V_0) \approx 5325$ GBP.

Problem 2

$$\begin{aligned}V_1 &\approx 1000\Big(S_1 + p(S_0) + \frac{\partial}{\partial S_0}p(S_0)(S_1 - S_0)\Big)\\ &\approx 1000\Big(S_0 + p(S_0) + \Big(1 + \frac{\partial}{\partial S_0}p(S_0)\Big)S_0 X\Big)\\ &= a + bX.\end{aligned}$$

Therefore, using that $a = V_0$ and $b \approx 121,647.7$,

$$\begin{aligned}\mathrm{VaR}_{0.01}(V_1 - V_0) &\approx V_0 - a + bF^{-1}_{-X}(0.99)\\ &= -bF^{-1}_X(0.01)\\ &= -\Big(1 + \frac{\partial}{\partial S_0}p(S_0)\Big)S_0\sigma_X t^{-1}_\nu(0.01)\}\\ &\approx 6473 \text{ GBP}.\end{aligned}$$

Problem 3

$$\begin{aligned}V_1 &\approx 1000\Big(S_1 + p(S_0,\sigma_0) + \frac{\partial}{\partial S_0}p(S_0,\sigma_0)(S_1 - S_0) + \frac{\partial}{\partial \sigma_0}p(S_0,\sigma_0)(\sigma_1 - \sigma_0)\Big)\\ &\approx 1000\Big(S_0 + p(S_0,\sigma_0) + \Big(1 + \frac{\partial}{\partial S_0}p(S_0,\sigma_0)\Big)S_0 X + \frac{\partial}{\partial \sigma_0}p(S_0,\sigma_0)\sigma_0 Y\Big)\\ &= a + bX + cY,\end{aligned}$$

where X and Y are the log-returns of the stock and implied volatility, respectively. Notice that $a = V_0$, $b \approx 121,647.7$, and $c \approx 6,828.73$

$$bX + cY \stackrel{\mathrm{d}}{=} \Big(b^2 \cdot 0.014^2 + c^2 0.015^2(1.9/3.9)\Big)^{1/2} Z,$$

where Z has a standard Student's t distribution with 3.9 degrees of freedom. Therefore

$$\begin{aligned}\mathrm{VaR}_{0.01}(V_1 - V_0) &\approx -\Big(b^2 \cdot 0.014^2 + c^2 0.015^2(1.9/3.9)\Big)^{1/2} t_{3.9}^{-1}(0.01)\\ &\approx 6478 \text{ GBP}.\end{aligned}$$

Problem 4

$$\begin{aligned}V_1 &= (E_0 \exp\{Y\})^{-1} 1000(S_0 \exp\{X\} + p(S_0 \exp\{X\}))\\ &\approx \frac{1000}{E_0}\Big((1-Y)(S_0 + p(S_0)) + \Big(1 + \frac{\partial}{\partial S_0} p(S_0)\Big) S_0 X\Big)\end{aligned}$$

and

$$\begin{aligned}L = -(V_1 - V_0) &\approx \frac{1000}{E_0}\Big((S_0 + p(S_0))Y - \Big(1 + \frac{\partial}{\partial S_0} p(S_0)\Big) S_0 X\Big)\\ &= 2017130Y - 1291377X\\ &\approx 1291377(1.562Y - X)\end{aligned}$$

The empirical estimator is $L_{[np]+1,n}$. Here $n = 248$ and $p = 0.01$ and $[np] + 1 = 3$. The empirical estimate is $l_{3,248}$. The log-return sample in the lower left plot in Figure 1 gives

$$\begin{aligned}l_{1,248} &\approx 1291377(1.562(-0.005) - (-0.075)) \approx 86768\\ l_{2,248} &\approx 1291377(1.562(0.002) - (-0.054)) \approx 73769\\ l_{3,248} &\approx 1291377(1.562(0.006) - (-0.045)) \approx 70215.\end{aligned}$$

We find that the empirical estimate of $\mathrm{VaR}_{0.01}(V_1 - V_0)$ is approximately 70215 SEK.

Problem 5

X has the Pareto distribution, $\mathrm{P}(X \le x) = 1 - x^{-2}$, $x > 1$. Y has the Exponential distribution $\mathrm{P}(X \le x) = 1 - e^{-x/4}$. $\mathrm{P}(X + Y > 15) \approx 23/500 = 0.046$ (empirical estimate). $\mathrm{P}(X + Y > 40) \approx \mathrm{P}(X > 40) \approx 0.000625$. Notice that

$$\begin{aligned}\mathrm{P}(X + Y > 40) &> \mathrm{P}(X > 40 \text{ or } Y > 40)\\ &= \mathrm{P}(X > 40) + \mathrm{P}(Y > 40) - \mathrm{P}(X > 40)\,\mathrm{P}(Y > 40)\\ &= 0.0006703716 \approx \mathrm{P}(X > 40)\end{aligned}$$

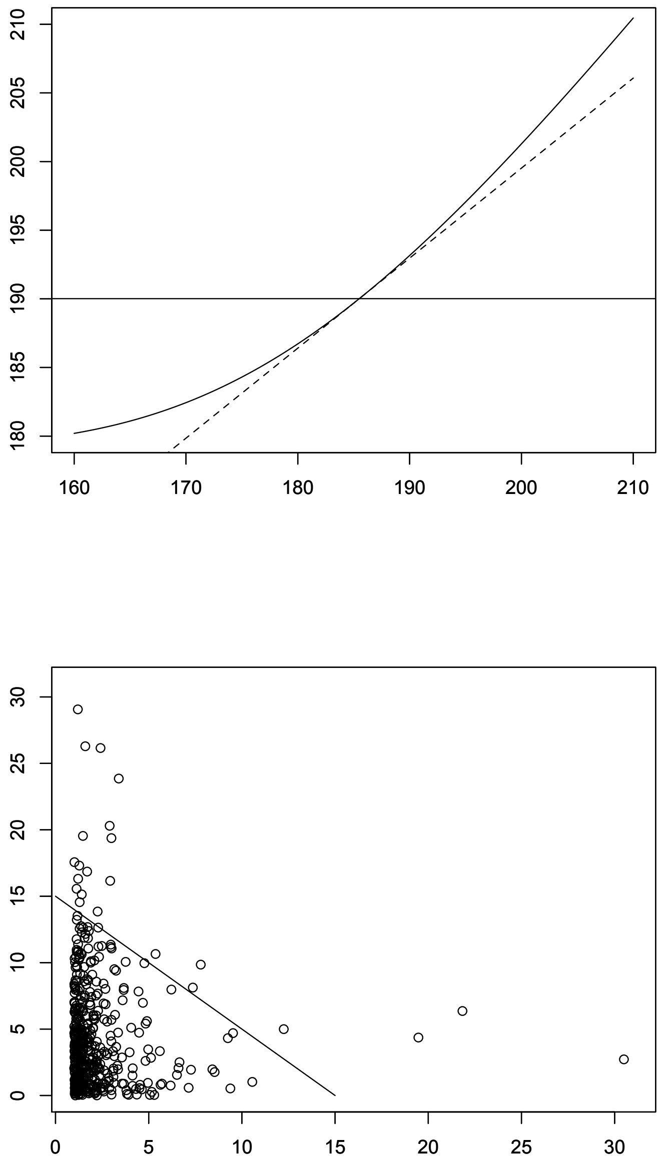

Figure 2: Upper plot: the portfolio value in Problems 1 and 2 as functions of the share price - with and without linearization. Lower plot: the fraction of observations satisfying $x + y > 15$ is the empirical estimate in Problem 5(a)